An Investigation of a Method of Balancing Inertial Loads On a Slider - Crank Mechanism

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Abstract

A machine is an assemblage of resisting bodies so arranged that it transforms motion in order to compel work. The transmission of power is being affected by the irregularities of motion due to the excess inertial loads. These are being affected by the irregularities of motion due to the excess inertial loads. These are being manifested by vibration, noise, and even fatigue problems which lead to destruction and failure.

The problem of extra motion can be corrected through mass balancing. In this study, two methods of balancing the excess inertial loads; specifically on a Slider-Crank mechanism, was being investigated.

The Chiou and Davies Method determines the shaking forces and moments and locates the position of the contra-rotating masses pivoted at different location to attain equilibrium on the system. The basis of location was done through analytical method using vectors and complex-algebra at different speeds of the crank.

The point-mass method of balancing put equivalent masses on its linkages to be at equilibrium. The putting of masses balances the system dynamically through the kinetic analysis. The two methods were being validated through numerical examples and the result was convincing.

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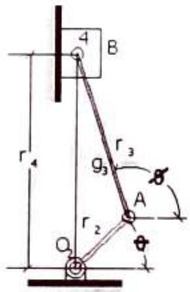
The slider- crank mechanism

A mechanism which translates a rotational motion into a translational one, as in reciprocating pumps and air compressors, where an electric motor drives the crank which in turn drives the piston that compresses the fluid. In some cases the slider is used as the input link and the crank as the output link. In this set-up, the mechanism transfers translational motion into rotary motion, as for instance in an internal combustion engine. Fig.1 is an illustration of a slider – crank mechanism.

Fig.1. The Slider – Crank Mechanism



r_2 = length of the crank
r_3 = length of the connecting rod
r_4 = instantaneous location of the piston from O_2
g_3 = location of the centroid of the connecting rod

Balancing the slider - crank mechanism

The purpose of balancing is to mitigate the shaking action of the inertia force associated with the extra movement of the masses. When masses move they develop forces which tend to destabilize the system, as in the rotation and reciprocation of masses in the slider –crank mechanism. Two different approaches of balancing will be considered here to reduce the out-of-balance forces which result to the shaking action. First, we consider the ideal location for the contra-rotating shafts with the balancing masses as presented by the Chiou and Davies methods and second, a point - mass system where the connecting rod is replaced by two kinetically equivalent point masses. One of the point mass is lumped with the piston and the other to the crank pin where they constitute the reciprocating and rotating masses, respectively. The reaction at the pivot point O_2 is the resultant of all the inertia forces acting on the system and where it is balanced by a pair of contra-rotating masses rotating at the appropriate radius properly proportioned based from the lengths of the crank and the connecting rod. The results will be compared based on Chiou and Davies methods on the different orders of the inertia forces, as they vary harmonically at frequencies equal to crank speed, twice the crank speed, four times the crank speed, respectively.

Computation for the instantaneous values of the shaking force and moments requires kinematic and kinetic analyses of the system. All other forces, such as weight and gas pressure, are left out of consideration.

Kinematic analysis of the slider - crank mechanism

From Fig. 1 , it is assumed that individual links are to be rigid bodies in which the distance between two given points remain fixed. But the instantaneous distance of the piston from the pivot point varies according to the expression

$$\begin{aligned} r_4 &= r_2 \sin\theta + r_3 \sin(180 - \varphi) \\ &= r_2 \sin\theta + r_3 \sin\varphi, \end{aligned} \qquad (1)$$

where

θ = the angle subtended by the crank to the horizontal axis

φ = the angle subtended by the connecting rod to the horizontal axis

Likewise, we can have the relationship of θ and φ given as

$$\cos\varphi = -\kappa \cos\theta , \qquad (2)$$

where

$$\kappa = \frac{r_2}{r_3}.$$

From the trigonometric identity, eq.(1) can be written in terms of a positive radical, because r_3 is costrained to pivot only at point A from $0 < \varphi < \pi$,

$$r_4 = r_2 \sin\theta + r_3 (1 - \cos^2 \varphi)^{1/2} \tag{3}$$

Substituting eq.(2) and by the binomial expansion for the expression $(1 - \cos^2 \varphi)^{1/2}$, we can rewrite eq. (3) neglecting higher order terms as

$$r_4 = r_2 \sin\theta + r_3 \left\{1 - \frac{1}{2}\kappa^2 \cos^2\theta - \frac{1}{8}\kappa^4 \cos^4\theta\right\} \tag{4}$$

The expansion for $\sin\varphi$ has its greatest error when $\cos^2\varphi$ takes its largest value. This occurs when $\cos\theta = \pm 1$ so that $\cos\varphi = \mu\kappa$. Hence,

$$\left(1 - \kappa^2\right)^{1/2} = 1 - \frac{1}{2}\kappa^2 - \frac{1}{8}\kappa^4$$

For the typical case of $\kappa = 1/3$,

LHS = 0.9428 and RHS = 0.9429

Making use of trigonometric identities which are given below without proof. Their derivation is left as an exercise in the use of complex variable notation.

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta) \tag{5.1}$$

$$\cos^3 = \frac{1}{4}(3\cos\theta + \cos 3\theta) \tag{5.2}$$

$$\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta) \tag{5.3}$$

$$\cos^5\theta = \frac{1}{16}(10\cos\theta + 5\cos 3\theta + \cos 5\theta) \tag{5.4}$$

$$\cos^6\theta = \frac{1}{32}(10 + 15\cos 2\theta + 6\cos 4\theta + \cos 6\theta) \tag{5.5}$$

so that eq. (4) can be written approximately as

$$r_4 = r_2 \sin\theta + r_3\left\{1 - \frac{1}{4}\kappa^2\left(1 + \frac{3}{16}\kappa^2\right) - \frac{1}{4}\kappa^2\left(1 + \frac{1}{4}\kappa^2\right)\cos 2\theta - \frac{1}{64}\kappa^2 \cos 4\theta\right\} \quad (6)$$

By differentiating eq.(6) twice with respect to time, we get approximately

$$\frac{d^2 r_4}{dt^2} = \ddot{r}_4 = \omega^2\left\{-r_2 \sin\theta + r_3\kappa^2\left(1 + \frac{1}{4}\kappa^2\right)\cos 2\theta + \frac{1}{4}r_3\kappa^4 \cos 4\theta\right\} \quad (7)$$

where,

$$\omega = d\theta / dt$$

and

$\frac{d^2\theta}{dt} = 0$, assuming the crank to rotate at constant angular speed.

Let,

$$P = -r_2\omega^2 \quad (7.1)$$

$$Q = r_3\omega^2\kappa^2\left(1 + \frac{1}{4}\kappa^2\right) \quad (7.2)$$

$$R = \frac{1}{4}r_3\omega^2\kappa^4, \quad (7.3)$$

so that eq. (7) can be rewritten approximately as

$$\ddot{r}_4 = P\sin\theta + Q\cos 2\theta + R\cos 4\theta. \quad (8)$$

This is the magnitude of the acceleration of the piston representing the different order of frequencies. In vector form we have

$$\bar{r}_4 = \ddot{r}_4 j \quad (9)$$

From Fig.1, each link can be represented in a vector form plotted as

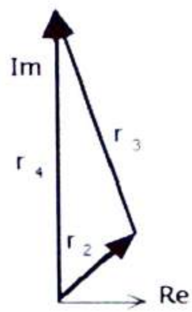


Fig.2 Vector representation of slider- crank mechanism

The loop closure equation of the vector polygon in Fig.2 is

$$\mathbf{r}_4 = \mathbf{r}_2 + \mathbf{r}_3 \quad (10)$$

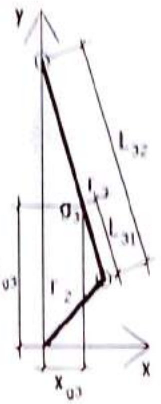
Differentiating eq.(10) twice with respect to time we get

$$\ddot{r}_4 = \ddot{r}_2 + \ddot{r}_3 \quad (11)$$

Solving for $\ddot{r}_3$ from eq.(11) , we have

$$\ddot{r}_3 = \ddot{r}_4 - \ddot{r}_2 \quad (12)$$

Fig.3 Location of the centroid on the rod



Knowingthe location of the centroid on the connecting rod as shown in Fig.3, we can get the acceleration of that point as

$$A_{g3} = \left[\frac{L_{31}}{L_{32}}\right](\ddot{r}_4 - \ddot{r}_2) + \ddot{r}_2 \qquad (13)$$

where

L_{31}= location of the centroid from the crank pin
L_{32}= r_3

Simplifying eq.(13), we have

$$A_{g3} = \left[\frac{L_{31}}{L_{32}}\right]\ddot{r}_4 + \left[\frac{L_{32} - L_{31}}{L_{32}}\right]\ddot{r}_2 \,, \qquad (14)$$

where

$$\begin{aligned}\ddot{r}_2 &= -r_2\omega^2(\cos\theta + j\sin\theta)\\ &= \mathrm{P}\ e^{j\theta}. \qquad (15)\end{aligned}$$

The acceleration A_{g3} at the centroid is used to calculate the inertia force which is responsible for the motion of the mechanism. This inertia force which is sometimes known as the ' fictitious force ' is the driving force which is necessary to maintain the motion and transmitted as forces on the bearings. The inertia effects due to the motion of a rigid body is equivalent to a force (- mA) acting through the centroid and a moment $(- I\ddot{\varphi})$ where $\ddot{\varphi}$ is the angular acceleration of the body. To determine a well-balanced system , it is necessary to evaluate the inertia force and moment for each of the members of the mechanism throughout the range of movement which will result to zero resultant force (which is the shaking force) .

The first approach of balancing

Chiou and Davies developed two methods of finding the ideal locations for contra-rotating shafts in order to reduce the effects of shaking force ad moment. The shaking force at a certain order of frequency can be represented by two, generally unequal, contra- rotating force vectors of constant magnitudes. They are produced by counterweights installed on shafts rotating at a multiple of the speed of the crank, thereby neutralizing the out-of-balance forces in the system.

From Fig.1, we can sum the inertia forces as

$$\sum F = m_3 A_{g3} + m_4 \ddot{r}_4 \tag{16}$$

The inertia force at link 2 is not considered because it can easily be balanced by a counterweight installed directly opposite to the crank.

So,the shaking force can then be expressed as

$$F_s = -\left(m_3 A_{g3} + m_4 \ddot{r}_4\right) \tag{17}$$

Substituting eq.(14) into eq.(17) and grouping like terms, we have

$$F_s = -\left\{\left(m_3\left[\frac{L_{31}}{L_{32}}\right] + m_4\right)\ddot{r}_4 + m_3\left[\frac{L_{32} - L_{31}}{L_{32}}\right]\ddot{r}_2\right\}. \qquad (18)$$

By substituting eqs. (8) and (15) into eq.(18) and rearranging terms, this will give us an expression for the shaking force in terms of function of certain frequencies, written as

$$F_s = TP\cos\theta + (T + S)Pj\sin\theta + SQj\cos 2\theta + SRj\cos 4\theta \ , \qquad (19)$$

where

$$S = -(m_3\left[\frac{L_{31}}{L_{32}}\right] + m_4) \qquad (20)$$

$$T = -m_3\left[\frac{L_{32} - L_{31}}{L_{32}}\right] \qquad (21)$$

m_3 = mass of the connecting rod

m_4 = mass of the piston

Hence the shaking force that varies harmonically at frequency equal to the crank speed is

$$F_{s1} = TP\cos\theta + j(T + S)P\sin\theta \ , \qquad (22)$$

at twice the crank speed is

$$F_{s2} = jSQ\cos 2\theta \ , \qquad (23)$$

and at four times the crank speed is

$$F_{s4} = jSR\cos 4\theta \ . \qquad (24)$$

In general, the shaking force components can be expressed in complex polar form, written as

$$F_{sk} = A\cos k\theta + Bj\sin k\theta \qquad (25)$$

and can be represented by two contra-rotating force vectors, expressed in exponential form as

$$F_{sk} = F_k^{\ +} e^{jk\theta} + F_k^{\ -} e^{-jk\theta} \qquad (26)$$

By comparing eqs. (25) and (26) and solving algebraically the expressions of F_k^+ and F_k^- as functions of A and B, we get

$$F_k^{\ +} = (A+B)/2 \qquad (27)$$

and

$$F_k^{\ -} = (A-B)/2 \ . \qquad (28)$$

For k=1,

$$A = TP \qquad (29)$$
$$B = (T+S)P \qquad (30)$$

so that ,

$$F_1^{+} = (2T + S)P/2$$
$$= \left\{2\left(-m_3\left[\frac{L_{32}-L_{31}}{L_{32}}\right]\right) - \left(m_3\left[\frac{L_{31}}{L_{32}}\right] + m_4\right)\right\}\left(\frac{-r_2\omega^2}{2}\right)$$
$$= \left\{\frac{1}{2}\left(m_3\left[\frac{L_{31}}{L_{32}}\right] + m_4\right) + m_3\left[\frac{L_{32}-L_{31}}{L_{32}}\right]\right\}(r_2\omega^2) \qquad (31)$$

and

$$F_{1^-} = -SP/2$$

$$= -\left(m_3 \left[\frac{L_{31}}{L_{32}} \right] + m_4 \right) \left(\frac{r_2 \omega^2}{2} \right) \qquad (32)$$

If we have a second look at eqs. (23) and (24) and compare this to eq.(25), the real term becomes the imaginary and the imaginary becomes a negative real term. The physical meaning is that the position of line of action of the shaking force advances by 90 ^{0} of rotation. So, we have a new equation for the shaking force written as,

$$F_s = jA\cos k\theta - B\sin k\theta \qquad (33)$$

and the contra-rotating force vectors, we have

$$F_s = jF_k^{\ +} e^{jk\theta} + jF_k^{\ -} e^{j(-k\theta)} \qquad (34)$$

$$= jF_k^{\ +} e^{j(k\theta+90)} + jF_k^{\ -} e^{j(-k\theta+90)} \qquad (35)$$

So, following the same algebraic processes for solving the two contra-rotating force vectors, we have

For k=2,

$$A = SQ \qquad (36)$$
$$B = 0 \qquad (37)$$

so that ,

$$F_2^+ = SQ/2$$

$$= -\frac{1}{2}\left(m_3\left[\frac{L_{31}}{L_{32}}\right] + m_4 \right)\left(r_3\omega^2\kappa^2\left[1 + \frac{1}{4}\kappa^2\right]\right) \qquad (38)$$

$$= F_2^-$$

For k=4,

$$A = SR \qquad (39)$$

$$B = 0 \qquad (40)$$

so that ,

$$F_4^+ = SR/2$$

$$= -\frac{1}{2}\left(m_3\left[\frac{L_{31}}{L_{32}}\right] + m_4 \right)\left(\frac{1}{4} r_3\omega^2\kappa^4 \right) \qquad (41)$$

$$= F_4^-$$

The shaking moment can be expressed as the summation of all the moments associated with the inertia forces passing through its centroid, written as

$$M_k = -\left(- m_3 y_3 A_{g3x} + m_3 x_3 A_{g3y} + I\ddot{\varphi}\right) \qquad (42)$$

and simplifying , we have

$$M_k = m_3 y_3 A_{g3x} - m_3 x_3 A_{g3y} - I\ddot{\varphi} \qquad (43)$$

where from Fig. 3 , x_3 and y_3 can be determined as

$$\begin{aligned} x_3 &= (L_{32} - L_{31})\sin(\varphi - \pi/2) \\ &= -(L_{32} - L_{31})\cos\varphi \\ &= \kappa(L_{32} - L_{31})\cos\theta \\ &= r_2\left[\frac{(L_{32} - L_{31})}{L_{32}}\right]\cos\theta \end{aligned} \qquad (44)$$

and

$$y_3 = r_2 \sin\theta + L_{31}\sin\varphi \qquad (45)$$

where $\sin\varphi$ can be expressed using binomial expansion and eq.(45) can be written as

$$y_3 = r_2 \sin\theta + L_{31}\left(1 - \tfrac{1}{2}\kappa^2\cos^2\theta - \tfrac{1}{8}\kappa^4\cos^4\theta\right) \qquad (46)$$

or in the order of different frequencies , x_3 and y_3 can be expressed as

$$x_3 = G\cos\theta \qquad (47)$$

and

$$y_3 = D + C\sin\theta + E\cos 2\theta + F\cos 4\theta \qquad (48)$$

where,

$$\begin{aligned} C &= r_3 \\ D &= L_{31}\left(1 - \tfrac{1}{4}\kappa^2 - \tfrac{3}{64}\kappa^4\right) \\ E &= -\tfrac{1}{4}L_{31}\kappa^2\left(1 + \tfrac{1}{4}\kappa^2\right) \\ F &= -\tfrac{1}{64}L_{31}\kappa^4 \\ G &= r_2\left[\frac{L_{32} - L_{31}}{L_{32}}\right] \end{aligned}$$

By expanding eq.(14) , writing down the components of the centroidal acceleration of the rod as

$$A_{g3x} = H\cos\theta \quad (49)$$

and

$$A_{g3y} = U\sin\theta + V\cos 2\theta + W\cos 4\theta \quad (50)$$

where,

$$H = P\left[\frac{L_{32} - L_{31}}{L_{32}}\right]$$
$$U = P$$
$$V = Q\left[\frac{L_{31}}{L_{32}}\right]$$
$$W = R\left[\frac{L_{31}}{L_{32}}\right]$$

Finally, the variation of magnitude of the inertia moment which acts around the centroid of the rod is the result of its angular acceleration. Mewes[5] presented certain equation of motion for the expression of the angular acceleration. The law for the angular acceleration is established from the identity

$$\frac{d^2\varphi}{dt^2} = \cos\left(\frac{d^2\sin\varphi}{dt^2}\right) - \sin\varphi\left(\frac{d^2\cos\varphi}{dt^2}\right) \quad (51)$$

where the expression for $\sin\varphi$ and $\cos\varphi$ was already derived previously.

By differentiating the bracketed terms in eq.(7) , the $\sin\varphi$ expression , twice with respect to time , we get

$$\frac{d^2 \sin\varphi}{dt^2} = \omega^2\left[\kappa^2\left(1+\tfrac{1}{4}\kappa^2\right)\cos 2\theta + \tfrac{1}{4}\kappa^4 \cos 4\theta\right] \qquad (52)$$

and

$$\frac{d^2 \cos\varphi}{dt^2} = \kappa\omega^2 \cos\theta \qquad (53)$$

so that eq. (54) becomes

$$\begin{aligned}\frac{d^2\varphi}{dt^2} = \ddot{\varphi} = &-\kappa\cos\theta\left[\kappa^2\left(1+\tfrac{1}{4}\kappa^2\right)\cos 2\theta + \tfrac{1}{4}\kappa^4\cos 4\theta\right]\omega^2 \\ &-\left[1-\tfrac{1}{4}\kappa^2\left(1+\tfrac{3}{16}\kappa^2\right)-\tfrac{1}{4}\kappa^2\left(1+\tfrac{1}{4}\kappa^2\right)\cos 2\theta - \tfrac{1}{64}\kappa^4\cos 4\theta\right]\kappa\omega^2\cos\theta\end{aligned} \qquad (54)$$

Making use of trigonometric identities given as

$$\begin{aligned}\cos m\theta\cos n\theta &= \tfrac{1}{2}\left(\cos\langle m+n\rangle\theta + \cos\langle n-m\rangle\theta\right)\\ \cos m\theta\sin n\theta &= \tfrac{1}{2}\left(\sin\langle m+n\rangle\theta + \sin\langle n-m\rangle\theta\right)\end{aligned}$$

eq. (56) can be written in the form

$$\ddot{\varphi} = a\cos\theta + b\cos 3\theta + c\cos 5\theta \qquad (55)$$

where, $a = -\left(\kappa + \tfrac{1}{8}\kappa^3 + \tfrac{3}{64}\kappa^5\right)\omega^2$

$b = -\left(\tfrac{3}{8}\kappa^3 + \tfrac{27}{128}\kappa^5\right)\omega^2$

$c = -\tfrac{15}{128}\kappa^5\omega^2$

and by substituting eqs.(44),(45),(49),& (50), into eq. (43)), we have the expression of the shaking moment as function of θ written as

$$M(\theta) = g\cos\theta + h\sin 2\theta + m\cos 3\theta + n\cos 5\theta \qquad (56)$$

where ,

$$g = \tfrac{1}{2}m_3\left[2DH + EH - \frac{L_{31}}{L_{32}}GQ\right] - Ia$$

$$h = \tfrac{1}{2}m_3\left[CH - GP\right]$$

$$m = \tfrac{1}{2}m_3\left[EH - \left(\frac{L_{31}}{L_{32}}\right)GQ - \left(\frac{L_{31}}{L_{32}}\right)GR + FH\right] - Ib$$

$$n = \tfrac{1}{2}m_3\left[FH - \left(\frac{L_{31}}{L_{32}}\right)GR\right] - Ic$$

At certain order of frequency, the shaking moment can be expressed in a vector function as

$$M(\theta) = M_k^{\,+}e^{jk\theta} + M_k^{\,-}e^{-jk\theta} \qquad (57)$$

This resultant moment vector, like the shaking force vector of a reciprocating mass, has constant direction and generally generates a straight line hodograph along the z-axis. If this shaking moment is to be treated as complex vector where the imaginary part is absent, the vectors $M_k^{\,+}$ and $M_k^{\,-}$ will be symmetrically disposed on the z-axis expressed as the complex conjugate

$$M_k^{\,+} = R_k + I_k j \qquad (58)$$

and $$M_k^{\,-} = R_k - I_k j \qquad (59)$$

In a more simplified form, it can be written as

$$M(\theta) = M_k \cos(k\theta + \alpha_k) \qquad (60)$$

Where $M_k = \left(R_k{}^2 + I_k{}^2\right)^{1/2}$

and $\alpha_k = \tan^{-}\left({}^{I_k}\!/_{R_k} \right)$, the angle subtended by M_k to the z-axis.

Consequently, the vectors $M_k{}^+$ and $M_k{}^-$ having the same magnitude are symmetrically disposed along the z – axis which are also the locations of the real components while the imaginary components are situated to lines along the xy plane.

For instance at $k = 1$

$$M_1 = 2\left(R_1{}^2 + I_1{}^2\right)^{1/2} \qquad (61)$$

The magnitude of the real component of the instantaneous shaking moment can be picked out from eq. (55) as the coefficient of cosine term written as

$$R_1 = g$$
$$= (1/2)m_3\left[2DH + EH - \left(\frac{L_{31}}{L_{32}}\right)GQ\right] - Ia$$

But $I_1 = 0$, since there is no sine term present in eq.(55) at first order of frequency,

The instantaneous shaking moment can be expressed as

$$M_1 = 2R_1$$

So that, $\alpha_1 = \tan^{-}\left({}^{0}\!/_{R_k} \right)$

For $k = 2$ $\qquad M_2 = 2\left(R_2{}^2 + I_2{}^2\right)^{1/2} \qquad (62)$

The magnitude of the real component of the instantaneous shaking moment from eq. (55) is

$$R_2 = 0$$

And the imaginary component is

$$I_2 = h$$
$$= (1/2)m_3[CH - GP]$$

And the magnitude of the instantaneous shaking moment is

$$M_2 = 2h$$

So that, $\alpha_1 = \tan^{-1}\left(I_2 / 0 \right)$

<u>For $k = 3$</u> $M_3 = 2\left(R_3{}^2 + I_3{}^2\right)^{1/2}$, the magnitude of the (63) instantaneous shaking moment vector about the origin and

$$I_3 = 0$$

So that the magnitude of the instantaneous shaking moment is

$$M_3 = 2R_3$$
$$= 2m$$

So that, $\alpha_3 = \tan^{-1}\left(0 / R_3 \right)$

For <u>$k = 4$</u>

There is no instantaneous shaking moment vector for this order of frequency but only shaking forces. This can be verified from eq.(62)

For <u>$k = 5$</u> $M_5 = 2\left(R_5{}^2 + I_5{}^2\right)^{1/2}$,the magnitude of the (64) instantaneous shaking moment vector about the origin and

$$I_5 = 0$$

So that the magnitude of the instantaneous shaking moment is

$$M_5 = 2R_5$$
$$= 2n$$

So that, $$\alpha_5 = \tan^{-1}\left({}^{0}\!/_{R_5} \right)$$

Chiou and Davies methods of balancing

Chiou and Davies developed two methods of locating the positions of the two contra-rotating shafts with the corresponding counterweights which turn this into a force that balances the out–of–balance forces in the system.

The first method : By the solution of equations

By assuming any point within the system with coordinates either x_k^+, y_k^+ or x_k^-, y_k^-, the other point can be found respectively. For instance , let S_k^+ be the assumed point that locates the axis of the shaft that carries the counterweights (- F_k^+) and rotating in the same direction as the crankshaft and with the coordinate points of x_k^+, y_k^+. The other point can be found following this process

If x_k^+, y_k^+ are chosen, then

$$x_k^- = a_k \sin\lambda_k - b_k \cos\lambda_k \tag{65}$$

and

$$y_k^- = -a_k \cos\lambda_k - b_k \sin\lambda_k \tag{66}$$

where

$$a_k = \{M_k \cos\alpha_k - F_k^+ \lfloor x_k^+ \sin\lambda_k - y_k^+ \cos\lambda_k \rfloor\} / F_k^-$$

and

$$b_k = \{- M_k \cos\alpha_k - F_k^+ \lfloor x_k^+ \cos\lambda_k + y_k^+ \sin\lambda_k \rfloor\} / F_k^-$$

However, if x_k^-, y_k^- are being chosen, then the other point can be found by the following equations as follows

$$x_k^{\ +} = d_k \sin \lambda_k + e_k \cos \lambda_k \qquad (67)$$

and

$$y_k^{\ +} = -d_k \cos \lambda_k + e_k \sin \lambda_k \qquad (68)$$

where,

$$d_k = \{M_k \cos \alpha_k + F_k^{\ -}[-x_k^{\ -} \sin \lambda_k + y_k^{\ -} \cos \lambda_k]\}/ F_k^{\ +}$$

and

$$e_k = \{-M_k \sin \alpha_k + F_k^{\ -}[x_k^{\ +} \cos \lambda_k + y_k^{\ -} \sin \lambda_k]\}/ F_k^{\ +}$$

The accuracy of these found points can be checked by satisfying the condition that

$$x_k^{\ +} F_k^{\ +} \sin \theta_k^{\ +} - y_k^{\ +} F_k^{\ +} \cos \theta_k^{\ +} + x_k^{\ -} F_k^{\ -} \sin \theta_k^{\ -} - y_k^{\ -} F_k^{\ -} \cos \theta_k^{\ -} \qquad (69)$$
$$= M_k \cos(k\theta + \alpha_k)$$

where,

$$\theta_k^{\ +} = \lambda_k^{\ +} + k\theta$$

and $\theta_k^{\ -} = \lambda_k^{\ -} - k\theta$

<u>The second method : By first finding the location of C_k</u>

By locating first the invariant center C_k, where the two contra-rotating force vectors $F_k^{\ +}$ and $F_k^{\ -}$ are rotating at certain frequency, the corresponding coordinate points can be found by the following equations

$$x_k = M_k \left[\frac{F_k^{\ +} \sin(\lambda_k - \alpha_k) - F_k^{\ -} \sin(\lambda_k - \alpha_k)}{F_k^{\ +2} - F_k^{\ -2}} \right] \qquad (70)$$

and

$$y_k = -M_k \left[\frac{F_k^{\ +} \cos(\lambda_k - \alpha_k) - F_k^{\ -} \cos(\lambda_k - \alpha_k)}{F_k^{\ +2} - F_k^{\ -2}} \right] \qquad (71)$$

where θ can have any value.

After locating the coordinate points, the following procedural steps are being suggested to locate for the other point where the axis of the contra-rotating shaft passes.

Step 1. Draw a vector, either L_k^+ or L_k^-, form C_k to the chosen shaft axis whose location is S_k^+ or S_k^-

Step 2. Draw the x' axis through C_k at an angle λ_k from the horizontal.

Step 3. Measure η, the angle subtended by the chosen vector to the x' axis.

Step 4. Draw the second vector at an angle $-\eta$ to the x' axis and of length L_k^+ or L_k^- such that

$$\left(\frac{L_k^-}{L_k^+}\right) = \left(\frac{F_k^+}{F_k^-}\right)$$

The second location of either S_k^+ or S_k^- is located at the end of this vector. The accuracy of this process can be checked, if the summation of moments at points C_k would result to zero. It must satisfy the following conditions,

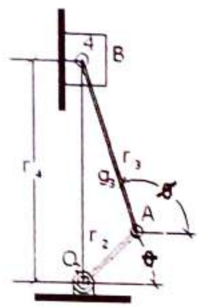
$$F_k^+ L_k^+ \cos\eta_k^+ - F_k^- L_k^- \cos\eta_k = 0 \qquad (72)$$

and

$$F_k^+ L_k^+ \sin\eta_k^+ + F_k^- L_k^- \sin\eta_k = 0 \qquad (73)$$

Numerical application

Consider a slider-crank mechanism with the following data :



- Crankshaft speed, ω=160 rad/sec,ccw
- Stroke= 13.8 cm
- Connecting rod weight, m_3= 1.5876 kg
- Piston weight m_4 = 1.134 kg
- Connecting rod length , r_3 =30.48 cm
- Connecting rod moment of inertia , I = 0.01243 N-s^2-m
- L_{31} = location of the centroid at link 3 from the crank pin, 8.89 cm
- $L_{32} = r_3$

Since the bodies are in motion like the reciprocation of the shaft, the rocking the connecting rod, and the rotation of the crank, the system will tend to shake or vibrate. Here, a method of balancing the out-of-balance forces will be shown with the methods developed by Chiou and Davies (1).

For k=1 , frequency equals to the crank speed

The first method: By the solution of equations

1. The balancing masses can be picked out from eqs. (31) and (32) expressed as

$$m_1^+ = (\tfrac{1}{2})[m_3[2L_{32} - L_{31}]/L_{32} + m_4]$$
= (1/2){1.5876[2(0.3048)-0.0889]/0.3048+1.134}
= 1.9231 kg

And the other mass is

$$m_1^- = -(½)[m_3[2L_{32} - L_{31}] + m_4]$$
$$= -(1/2)[1.5876(0.0889/0.3048)+1.134]$$
$$= -0.798525 \text{ kg}$$

2. The contra-rotating forces are

$$F_1^+ = m_1^+ r_2 \omega^2$$
$$= 1.9231(0.1016)(160)^2$$
$$= 5001.9062 \text{ N}$$

And for

$$F_1^- = m_1^- r_2 \omega^2$$
$$= -0.798525(0.1016)(160)^2$$
$$= -2076.9316 \text{ N}$$

3. At the start of the cycle, the shaking force is

$$F_1(0) = TP$$
$$= [m_3(L_{32} - L_{31})/L_{32}](-r_2\omega^2)$$
$$= -2924.9096 \text{ N}$$

or in exponential form

$$F_1(0) = 2193.682e^{j\lambda} \text{ N}$$

where,

$$\lambda_1 = 180 \text{ deg}$$

4. The magnitude of the instantaneous shaking moment is

$$M_1 = 2g$$
$$= 2(\tfrac{1}{2}m_3\left[2DH + EH - \frac{L_{31}}{L_{32}}GQ\right] - Ia)$$
$$= -312.482835 \text{ Nm}$$

and

$$\alpha_1 = \tan^{-1}\left(0/-312.482835\right) = 180 \text{ deg}$$

5. Choose a point where the axis of the shaft that carries the counterweights will be located. In this case, it is the pivot point of the crank. So, the value of our $x_1^+ = 0$ and $y_1^+ = 0$. To locate the other point, using the derived equation , we have

$$x_1^- = -0.150454\sin(180) - 0$$
$$= 0$$

and

$$y_1^- = -(-0.150454)\cos(180) - 0$$
$$= -0.15054 \text{ m}$$

<u>*To check*</u>

$$x_1^+ F_1^+ \sin\theta_1^+ - y_1^+ F_1^+ \cos\theta_1^+ + x_1^- F_1^- \sin\theta_1^- - y_1^- F_1^- \cos\theta_1^-$$
$$= M_1 \cos(\theta + \alpha_1)$$

where,

$$\theta_1^- = \lambda_1 - k\theta$$

So,

0–0+ 0 – (-0.150454)(-2076.9316)cos(180-60) =-312.48283cos(60+180)
156.241333 = 156.241417

*Difference = -8.45 x 10^-5^*

The second method: by first finding the location of C_k

6. To locate the position of C_1, we have to solve for its coordinate points as

$$x_1 = M_1\left[\frac{F_1^{+}\sin(\lambda_1-\alpha_1)-F_1^{-}\sin(\lambda_1-\alpha_1)}{F_1^{+2}-F_1^{-2}}\right]$$

= {-312.482835[5001.9062sin(0-180)-(-2076.9316sin(0-180))]}/ $[(5001.9062)^2 - (2076.9316)^2]$

= 0

and for

$$y_1 = -M_1\left[\frac{F_1^{+}\cos(\lambda_1-\alpha_1)-F_1^{-}\cos(\lambda_1-\alpha_1)}{F_1^{+2}-F_1^{-2}}\right]$$

=-{-312.482835[5001.9062cos(0-180)-(-2076.9316cos(0-180))]}/ $[(5001.9062)^2 - (2076.9316)^2]$

= 0.10683267 m

7. So the length of L_1^{+} can have a value measured from the origin (0,0) as

L_1^{+} = 0.10683267 m

8. The other length L_1^{-} where the end of it is the location of the contra-rotating shaft and can be determined through the equation of

L_1^{-} = 0.10683267(5001.9062)/(-2076.9316)

= -0.2578676 m

9. <u>To check :</u>

$$F_1^{+}L_1^{+}\cos(\eta_1^{+}) - F_1^{-}L_1^{-}\cos(\eta_1^{-}) = 0$$

Knowing that

$$\eta_1^{+} = 90^0$$

and $\eta_1^{-} = -90^0$

so $0 = 0$

and $F_1^{+}L_1^{+}\sin(\eta_1^{+}) + F_1^{-}L_1^{-}\sin(\eta_1^{-}) = 0$

Knowing that

$$\eta_1^{+} = 90^0$$

and $\eta_1^{-} = -90^0$

so that,

(5001.9062)(0.10683267)(1) – (-2076.9316)(-0.2578676)(-1) = $-7.67\text{x}10^{-6}$

For k = 2 , frequency equals to twice the crank speed

1. The two balancing masses are equal to

$$m_2^{+} = -(1/2)\left\langle m_3\left(L_{31}/L_{32}\right) + m_4\right\rangle\left[\kappa^2\left\{1 + \left(\tfrac{1}{4}\right)\kappa^2\right\}\right]$$
$$= -0.09612 \text{ kg}$$
$$= m_2^{-}$$

2. The contra-rotating forces are

$$F_2^{+} = -SQ/2$$
$$= \left(m_2^{+}\right)r_3\omega^2$$
$$= -750.0128 \text{ N}$$
$$= F_2^{-}$$

3. At the start of the cycle, the shaking force is

$$F_2(0) = SQj$$

where

$$S = -\left\{m_3\left(L_{31}/L_{32}\right) + m_4\right\}$$
$$Q = r_3\omega^2\kappa^2\left[1 + \left(\tfrac{1}{4}\right)\kappa^2\right]$$

or in exponential form

$$F_2(0) = SQe^{j\lambda} \text{ N}$$

where, $\lambda = 270\,\text{deg}$

4. The magnitude of the instantaneous shaking moment is

$$M_2 = 2\left(R_2{}^2 + I_2{}^2\right)^{1/2}$$

since,

$$R_2 = 0$$

$$I_2 = (1/2)m_3[CH - GP]$$

$$M_2 = \text{-}0.0137589 \text{ Nm}$$

and

$$\alpha_2 = \tan^{-1}\left(\frac{-0.0137589}{0}\right) = -90^0$$

This is almost a zero moment condition where the two contra-rotating forces are concurrent at the pivot of the crank.

For k = 3 , frequency equals to three times the crank speed

In this instance, the shaking force action is zero. However, there is still that shaking action produced by the instantaneous moment expressed as

$$\begin{aligned} M_3 &= 2m \\ &= 2\{(1/2)m_3[EH - GV - GW + FH] - Ib\} \\ &= m_3(H(E+F) - G(V+W)) \end{aligned}$$

$$M_3 = -15.576244 \text{ Nm}$$

and

$$\alpha_3 = \tan^{-1}\left(0/-15.576244\right) = 180 \text{ deg}$$

For k = 4 , frequency equals to four times the crank speed

1. The balancing masses are two contra-rotating weights equal to

$$m_4^+ = -(1/2)\left[m_3\left(L_{31}/L_{32}\right) + m_4\right](1/4)\kappa^{4}$$

$$= -2.4646\text{x}10^{-3} \text{ kg}$$

$$m_4^+ = m_4^-$$

2. The contra-rotating forces are

$$F_4^+ = SR/2$$
$$= \left(m_4^+\right)r_3\omega^2$$
$$= -19.23098 \text{ N}$$
$$= F_4^-$$

3. At the start of the cycle, the shaking force is

$$F_4(0) = SRj$$

where , $S = -\left\{m_3\left(L_{31}/L_{32}\right) + m_4\right\}$

and $R = (1/4)r_3\omega^2\kappa^{4}$

or in exponential form , we have

$$F_4(0) = S\,\mathrm{R}e^{j\lambda} \text{ N}$$

where $\lambda = 270$ deg

4. The magnitude of the instantaneous shaking moment is

$$M_4 = 0$$

The value of the instantaneous shaking moment is zero because at higher order of frequencies, the value of k at k >2 in the series at eq. (69) is

$$k = (2n-1)\ ,\ \ n = 2,3,\ldots$$

and it is expressed in odd order of frequencies and graphically, the zero moment is the result when the two contra-rotating forces are concurrent at certain reference point.

For k = 5, frequency equals to five times the crank speed

The shaking force action is not present at this order of frequency because the expression of the shaking force at eq.(19) is at the even order of frequency and graphically, this is the result when the two opposing co-planar forces are rotating in the same direction. The shaking moment action is expressed as

$$\begin{aligned} M_5 &= 2n \\ &= 2\{(1/2)m_3[FH - G] - Ic\} \end{aligned}$$

$$M_5 = -2.394346 \text{ Nm}$$

and

$$\alpha_5 = \tan^{-1}\left(0/_{-2.394346}\right) = 180 \text{ deg}$$

Balancing slider-crank mechanism using point masses system

Data:

Crankspeed , 160 rad/sec , ccw

m_3 = mass of link 3

m_4 = mass of the piston

r_3 = length of the connecting rod

r_4 = instantaneous distance of the piston from the connecting rod

In balancing a slider-crank mechanism, the usual take off point is balancing the reciprocating mass. But before we can consider our one-cylinder engine to be complete, we have to assign mass or weight to the crank and connecting rod. The former can easily be balanced by rotating counterweights on the opposite side of the shaft, thus making it simple and straight forward. The connecting rod is a matter of subtle importance. As a first approximation, we may consider to represent the connecting rod by two point masses. The distribution of these masses depend on the location of the centroid. One part is grouped in the rotating mass comprising the crank and part of the rod and the other part is grouped to the reciprocating mass comprising the piston and the other end of the rod.

The kinetically equivalent point masses of the connecting rod can be calculated , as referred from Fig. 3 , as follows

$$m_{B3} = m_3\left(\frac{L_{31}}{L_{32}}\right)$$

$$m_{A3} = m_3\left(\frac{L_{32} - L_{31}}{L_{32}}\right)$$

Since the connecting rod is the prime object in balancing the slider – crank mechanism, the total inertia forces for the reciprocation and rotation of masses on the connecting rod is expressed as

$$\sum F = \left(m_{B3} + m_4\right)\ddot{r}_4 + m_{A3}\ddot{r}_2$$

Where,

$\ddot{r}_4$ = the acceleration of the piston

$= \left[P\sin\theta + Q\cos 2\theta + R\cos 4\theta\right]j$

Also,

$\ddot{r}_2$ = the acceleration of the crank

$$= P[\cos\theta + j\sin\theta]$$

A. So to balance the system at the frequency which is at resonance with the crank speed, the shaking force at the pivot point is

$$F_s = -\sum F$$
$$= -[(m_{B3} + m_4)\ddot{r}_4 + m_{A3}\ddot{r}_2]$$

In general in complex polar form and at frequency equal to the crank speed, the shaking force can be expressed as

$$F_s = F_1^{+} e^{j\theta} + F_1^{-} e^{-j\theta}$$

Where F_1^{+} and F_1^{-} are the two contra-rotating forces developed from the added masses other than the mass of the crank and expressed as

$$F_1^{+} = \left\{(1/2)\left[m_3 \, {L_{31}}/{L_{32}} + m_4\right] + m_3(L_{32} - L_{31})/L_{32}\right\} r_2\omega^2$$

and

$$F_1^{-} = -(1/2)\{m_3(L_{32} - L_{31})/L_{32}\} r_2\omega^2$$

B. To balance the system at the frequency which is twice the crank speed , the shaking force at the pivot point is

$$F_s = -(m_{B3} + m_4)\ddot{r}_4$$
$$= -\left(m_3 \frac{L_{31}}{L_{32}} + m_4\right) Qj\cos 2\theta$$

and the two contra-rotating forces which represent the shaking force and the basis for our balancing are given as

$$F_2^{\ +} = -(\tfrac{1}{2})\left\{m_3\left(\tfrac{L_{31}}{L_{32}}\right) + m_4\right\}\kappa^2\left[1 + (\tfrac{1}{4})\kappa^2\right]r_3\omega^2$$

$$= F_2^{\ -}$$

So the shaking force is

$$F_s = F_2^{\ +} e^{j(2\theta+90)} + F_2^{\ -} e^{-(2\theta+90)}$$

C. To balance the system at the frequency which is four times the crank speed , the shaking force at the pivot point is

$$F_s = -(m_{B3} + m_4)\ddot{r}_4$$

$$= -\left(m_3 \frac{L_{31}}{L_{32}} + m_4\right)Rj\cos 4\theta$$

and the two contra-rotating forces which represent the shaking force and the basis for our balancing are given as

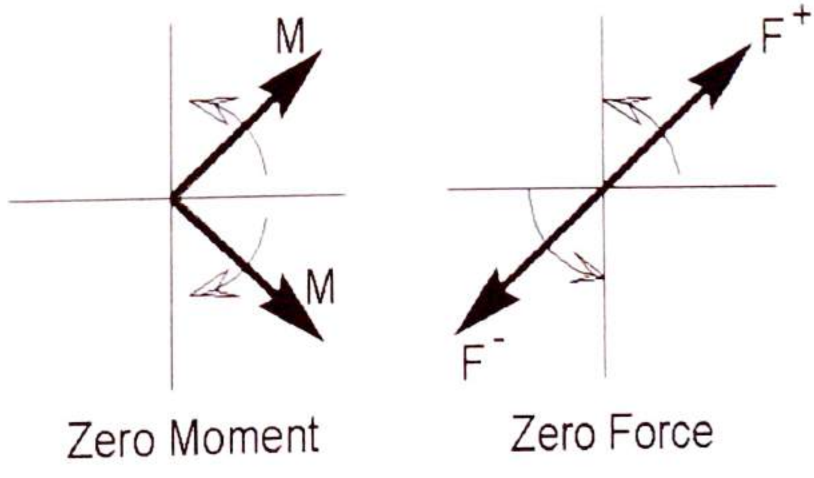
$$F_4^{\ +} = -(\tfrac{1}{8})\left\{m_3\left(\tfrac{L_{31}}{L_{32}}\right) + m_4\right\}\kappa^4 r_3\omega^2$$

$$= F_4^{\ -}$$

DISCUSSIONS

The two methods for balancing the slider – crank mechanism by Chiou and Davies, and the point masses system have some common aspects along the determination of the shaking force. The former determines the value of the shaking force through the summation of the inertia force for all moving bodies such as; the connecting rod and the piston, through their centroids. The crank is not included because it can easily be balanced through rotating counterweights installed opposite the crankshaft. However, the latter determines the shaking force by converting the connecting rod to a kinetically equivalent two point masses. Where the analyses are based on the reciprocation and rotation of the point masses, referred from their revolute joints. Analytically, their results turn out to be equal at three order of frequencies, specially on their shaking forces.

For the shaking moments at frequencies equal to twice and four times the crank speed, their values are zero. Graphically, the radii of these contra-rotating forces have their loci at the pivot point of the crank. With their frequencies equal to three and five times the crank speed, their shaking forces are zero meaning, their counterweights are positioned in such a way that their line of action is common while rotating in the same direction. See illustration below for graphical meaning of this phenomenon.



The results also show that as the order of frequency increases the balancing mass decreases and approaches to zero. The system reaches stability at a very high frequency and a tendency to vibrate when it slows down. So balancing should start at lower order of frequency in order to avoid unwanted motion of the mechanism.

Chiou and Davies furthermore, developed a method for determining the ideal location for the axes of contra-rotating shafts, where the counterweights are being installed or mounted. These shafts will balance the inertial force and moments which are responsible for the shaking action. But there are conditions where the locations would be impractical because of the need for another space for the location of this contra-rotating shaft. Meaning, that there's a need for a larger frame or space . However, for the point masses system, all of the contra-rotating shaft axes are located at the pivot point of the crank. The first order of frequency resonates with the crank speed and the rotating counterweights have radii equal to the crank. The second and fourth order of frequencies, have their radii being proportionated with the length of the connecting rod. The advantage of this method is, there is no need for a larger frame to balance the system. But a special linkages must be developed to synchronize the motion of the counterweights.

Recommendation

For further investigation of the methods developed by Chiou and Davies, an experimental rig with electronically actuated sensors which, can be interfaced to personal computers to show changes of motion at different frequencies should be assembled. If possible a vibration analyzer which is capable in evaluating the extra movement of the linkages should be considered for this purpose.

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