Determination of Thermal Diffusivity by Transient Analysis of Spherically Shaped Metal Specimen

ELISEO P. VILLANUEVA

Abstract

The partial differential equation that governs the conduction of heat is applied to a spherically shaped specimen. The solution to the equation was derived with the assumption that the surface heat resistance is negligible compared with the body heat resistance and that the body does not generate heat. The solution is an equation with time and thermal diffusivity as the independent variables and temperature as the dependent. The equation is to be used for the determination of thermal diffusivity with known values of time and temperature. A computer program was generated for the numerical calculation of the thermal diffusivity and its accuracy was investigated.

Introduction

When a solid body is suddenly immersed into an environment of different temperature, some time must elapse before an equilibrium temperature condition prevails in the body. The equilibrium condition is referred to as the steady state and the interim period before equilibrium is established as the transient heating or cooling process. In the transient or unsteady state, the analysis must take into account the change in

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ELISEO P. VILLANUEVA, Professor, College of Engineering, MSU-Iligan Institute of Technology, Iligan City, Philippines, has a Ph.D. in Mechanical Engineering (1999) from the University of New South Wales, Sydney, Australia and an M.S. in Nuclear Engineering (1984) from the University of the Philippines, Diliman, Quezon City, Philippines.

internal energy of the body with time and boundary conditions must be adjusted to match the physical situation that is apparent in the unsteady heat transfer problem. The property of a material that indicates its ability to change internal energy with time is thermal diffusivity. It is defined as thermal conductivity divided by the product of the specific heat and density. It reflects the ability of the material to conduct thermal energy compared to its ability to store thermal energy. Materials with a large thermal diffusivity respond quickly to changes in the thermal environment. Once the thermal diffusivity is determined, the thermal conductivity may be calculated after measuring the specific heat assuming that the density has been determined.

Objectives of the Study

The objectives of the study are the following:

1) to derive a solution to the general heat conduction equation of a case where the boundary conditions and assumptions can be possibly performed experimentally;

2) to generate a computer program that can numerically determine the thermal diffusivity from a set of temperature and time elapsed data obtained by transient heat transfer experiment; and

3) to evaluate the reliability and accuracy of the numerical calculation.

Review of Related Literature

The heat flow in a body of homogeneous material is governed by the heat transfer equation (Holman, 1997) in rectangular coordinates shown below:

$$\frac{1}{h^2}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\dot{q}}{k} \qquad (1)$$

where:

h^2 = thermal diffusivity
u = temperature
t = time
k = thermal conductivity
$\dot{q}$ = energy generated per unit volume

The above equation can be transformed into either in cylindrical or spherical coordinates form by standard calculus techniques (Wylie, 1982). Solutions to the conduction heat transfer partial differential equation may be solved in situations where appropriate boundary and initial conditions are applied. The numerical evaluation of the solutions is done by employing available numerical methods (Chapra, 1998).

Analysis

Equation 1 is applied to a situation where a body at a certain temperature is suddenly immersed in an environment of different temperature. The temperature variation with respect to time is then determined.

In solving equation 1 the following assumptions are applied:

1) the shape of the body to be analyzed is spherical with radius R;
2) the body is made of a homogeneous material;
3) the temperature of the surrounding environment does not change with time;
4) the surface heat resistance is negligible compared with the body heat resistance;
5) the body does not generate heat; and
6) the body is initially at a uniform temperature

The above assumptions are based on the premise that they can be possibly applied in an experimental set-up.

Equation 1 is solved by first transforming it into a spherical coordinates form. The transformed equation is shown below:

$$\frac{1}{h^2}\frac{\partial u}{\partial t}=\frac{1}{r}\frac{\partial^2(ru)}{\partial r^2}+\frac{1}{r^2\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial u}{\partial\theta}\right)}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}+\frac{\dot{q}}{k} \qquad (2)$$

Applying the above assumptions, equation 2 after rearranging becomes

$$\frac{\partial u}{\partial t}=h^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}\right) \quad \text{, for } 0 < r < R \qquad (3)$$

The boundary and initial conditions are

a) as $r \to R^-$, $u \to u_1$, for $0 < t$
b) as $t \to 0^+$, $u \to u_0$, for $0 \le r \le R$

The above conditions can be translated to another reference point by letting $u = u - u_1$ resulting to

a) as $r \to R^-$, $u \to 0$, for $0 < t$
b) as $t \to 0^+$, $u \to u_0 - u_1$, for $0 \le r \le R$

Equation 3 can be transformed into a simpler form by changing the dependent variable to

$$u=\frac{v}{r}$$

Substituting it into equation 3 with $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial r}-\frac{1}{r^2}v$ and $\frac{\partial^2 u}{\partial r^2}=\frac{1}{r}\frac{\partial^2 v}{\partial r^2}-\frac{2}{r^2}\frac{\partial v}{\partial r}+\frac{2}{r^3}v$

and after simplifying would result to

$$\frac{\partial v}{\partial t}=h^2\frac{\partial^2 v}{\partial r^2}$$

(4)

where the boundary and initial conditions are

a) as $r \to R^-$, $v \to 0$, for $0 < t$
b) as $t \to 0^+$, $v \to u_0 r - u_1 r$, for $0 \le r \le R$

By the method of separation of variables

$$v = exp\left(-h^2\beta^2 t\right)\left[A\cos\beta r + B\sin\beta r\right]$$

(5)

When r = 0, $0 = \exp\left(-h^2\beta^2 t\right)\left[A + 0\right]$; thus A=0

Then $v = \exp\left(-h^2\beta^2 t\right)B\sin\beta\rho$

Applying the boundary conditions; when $r \to R^-$, $v \to 0$

$0 = B\sin\beta r\exp\left(-h^2\beta^2 t\right)$, $\sin\beta r = 0$

$\beta = \dfrac{n\pi}{R}$, n = 1,2,3,4,....................

Substituting β would result to

$$v = \sum_{n=1}^{\infty}\exp\left[-\left(\frac{hn\pi}{R}\right)^2 t\right]B_n\sin\left(\frac{n\pi}{R}r\right)$$

Applying the initial condition, $t \to 0^+$, $v \to u_0 r - u_1 r$ and using Fourier series

$$B_n = \frac{2}{R}\int_0^R (u_0 - u_1) r\left(\sin\frac{n\pi}{R}r\right)dr$$ then

$$B_n = \frac{2R(u_0 - u_1)}{n\pi}(-1)^{n+1}$$

Substituting into the equation for v and solving for u knowing that v=ur would result to

$$u = \frac{1}{r}\frac{2R(u_0 - u_1)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\exp\left[-\left(\frac{hn\pi}{R}\right)^2 t\right]\sin\frac{n\pi}{R}r$$

Since $u = u - u_1$ then the final form of the equation for u would be

$$u = u_1 + \frac{1}{r}\frac{2R(u_0 - u_1)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\exp\left[-\left(\frac{hn\pi}{R}\right)^2 t\right]\sin\frac{n\pi}{R}r \tag{6}$$

The temperature at the center of the sphere , i.e., r=0, can be determined by rearranging equation 6 and expanding the term $\sin\frac{n\pi}{R}r$, resulting to the following:

$$u = u_1 + \frac{2R(u_0 - u_1)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\exp\left[-\left(\frac{hn\pi}{R}\right)^2 t\right]\left[\frac{n\pi}{R} - \frac{1}{3!}\left(\frac{n\pi}{R}\right)^3 r^2 + \frac{1}{5!}\left(\frac{n\pi}{R}\right)^5 r^4 - \ldots\ldots\right]$$

At r = 0 the equation for the center temperature of the sphere at any time t will be

$$u_c = u_1 + 2(u_0 - u_1)\sum_{n=1}^{\infty}(-1)^{n+1}\exp\left[-\left(\frac{hn\pi}{R}\right)^2 t\right] \tag{7}$$

The above equation can be used in the determination of the thermal diffusivity h^2. A computer program applying the incremental and bracketing techniques in numerical calculation is generated in order to determine it.

The accuracy of the numerical calculation of the thermal diffusivity is done by first calculating the center temperature u_c given the values of the elapsed time t, thermal diffusivity h^2, initial temperature u_1, and surrounding temperature u1 using equation 7. With the obtained values of the center temperatures and the elapsed time and using the same values of the other variables, the values of the thermal diffusivity are then calculated applying the numerical techniques. The results of the numerical calculation are then compared with the values of the thermal diffusivity used in the determination of the center temperatures. Theoretically, the values should be equal. If there are differences, then, they can be attributed to errors in the numerical calculation. The errors in percentage are determined as the calculated value minus the exact value divided by the exact value times one hundred.

Results and Discussion

The calculation of the center temperatures is based on the following data:

Radius of sphere = .02 m
Initial temperature of sphere = 30 °C
Surrounding temperature = 200 °C

The materials to be used are aluminum (pure), cast iron (4% C), and stainless steel (AISI 302) that have values of the thermal diffusivity equal to 9.71 e-5 m^2/s, 1.67e-5 m^2/s, and 0.391e-5 m^2/s respectively (Hagen, 1999). The materials represent the relatively high to relatively low values of the thermal diffusivity of metals.

Tables 1 to 3 show the results of the calculation using equation 7 and the numerical method of determining the thermal diffusivity. The errors are also indicated. It can be observed that there are some calculated values which are not indicated. It means that the numerical calculation does not yield any result during the iteration process. The computer program is designed in such a way that, for a given acceptable accuracy, it will only iterate within a defined maximum number of iteration and will proceed to the next set of time and temperature data even if no result is obtained. The maximum number of iteration used is 10,000.

Figures 1 to 3 show the plots of the numerical calculation of the thermal diffusivity as compared with the exact values.

Table 1. *Results of the calculation using equation 7 and the numerical method of determining the thermal diffusivity for pure aluminum material.*

Time (sec.)	Center Temperature (°C)	Calculated Thermal Diffusivity (m^2/s)	Exact Thermal Diffusivity (m^2/s)	Error (%)
0.5	100.1975		9.71e-5	
1.0	169.0523	9.702246e-5	9.71e-5	-0.07986
1.5	190.6527	9.702246e-5	9.71e-5	-0.07986
2.0	197.1788	9.702246e-5	9.71e-5	-0.07986
2.5	199.1485	9.603996e-5	9.71e-5	-1.0917
3.0	199.7430	9.603996e-5	9.71e-5	-1.0917
3.5	199.9224	8.835998e-5	9.71e-5	-9.00105
4.0	199.9766	8.099999e-5	9.71e-5	-16.5809
4.5	199.9929	7.396001e-5	9.71e-5	-23.8311
5.0	199.9979	6.724001e-5	9.71e-5	-30.7518

Table 2. *Results of the calculation using equation 7 and the numerical method of determining the thermal diffusivity for cast iron (4%C) material.*

Time (sec.)	Center Temperature (°C)	Calculated Thermal Diffusivity (m^2/s)	Exact Thermal Diffusivity (m^2/s)	Error (%)
1.0	32.35539		1.67e-5	
2.0	63.25083	1.670765e-5	1.67e-5	0.045808
3.0	103.6503	1.670765e-5	1.67e-5	0.045808
4.0	135.055	1.670765e-5	1.67e-5	0.045808
5.0	156.7694	1.670765e-5	1.67e-5	0.045808
6.0	171.3271	1.670765e-5	1.67e-5	0.045808
7.0	181.0024	1.670765e-5	1.67e-5	0.045808
8.0	187.4167	1.670765e-5	1.67e-5	0.045808
9.0	191.666	1.670765e-5	1.67e-5	0.045808
10.0	194.4805	1.670765e-5	1.67e-5	0.045808
11.0	196.3445	1.670765e-5	1.67e-5	0.045808
12.0	197.579	1.680999e-5	1.67e-5	0.045808
13.0	198.3967	1.680999e-5	1.67e-5	0.045808
14.0	198.9381	1.680999e-5	1.67e-5	0.045808
15.0	199.2967	1.680999e-5	1.67e-5	0.045808

Table 3. Results of the calculation using equation 7 and the numerical method of determining the thermal diffusivity for stainless steel (AISI 302) material.

Time (sec.)	Center Temperature (°C)	Calculated Thermal Diffusivity (m^2/s)	Exact Thermal Diffusivity (m^2/s)	Error (%)
1.0	30.00003			
2.0	30.00389		3.91e-6	
3.0	30.2226		3.91e-6	
4.0	31.62183		3.91e-6	
5.0	35.21092		3.91e-6	
6.0	41.15746		3.91e-6	
7.0	48.99106		3.91e-6	
8.0	58.04765	3.912978e-6	3.91e-6	0.076164
9.0	67.72136	3.912978e-6	3.91e-6	0.076164
10.0	77.5469	3.912978e-6	3.91e-6	0.076164
11.0	87.2006	3.912978e-6	3.91e-6	0.076164
12.0	96.47401	3.912978e-6	3.91e-6	0.076164
13.0	105.2439	3.912978e-6	3.91e-6	0.076164
14.0	113.4464	3.912978e-6	3.91e-6	0.076164
15.0	121.058	3.912978e-6	3.91e-6	0.076164
16.0	128.0809	3.912978e-6	3.91e-6	0.076164
17.0	134.5338	3.912978e-6	3.91e-6	0.076164
18.0	140.4448	3.912978e-6	3.91e-6	0.076164
19.0	145.8474	3.912978e-6	3.91e-6	0.076164
20.0	150.7769	3.912978e-6	3.91e-6	0.076164

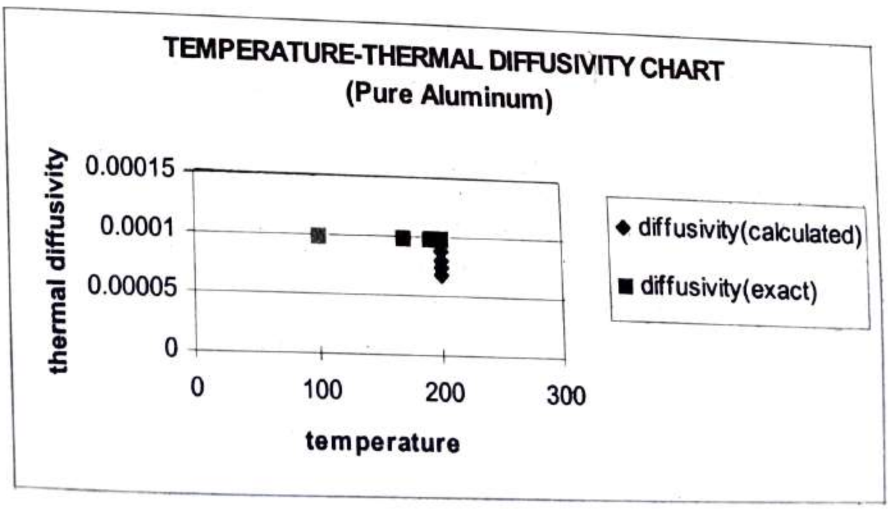

Figure 1.a. *Plot of the results for pure aluminum material.*

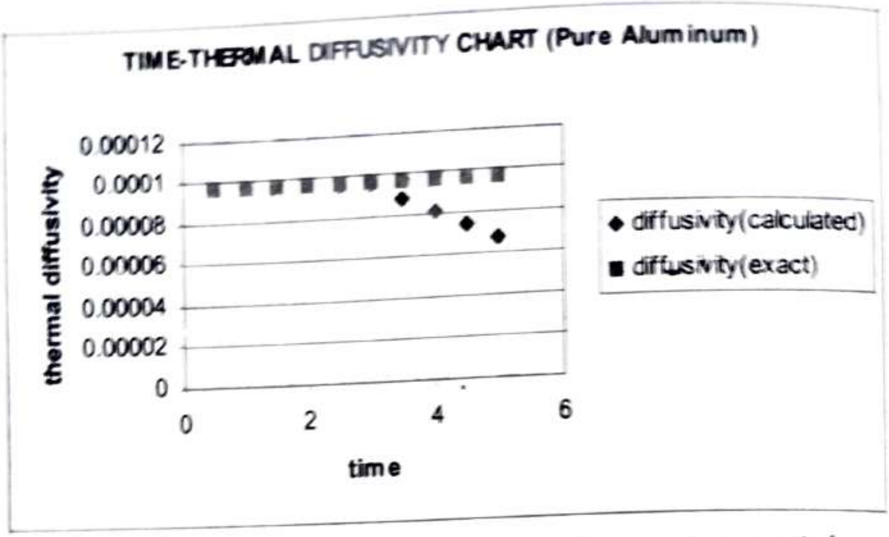

Figure 1.b. Plot of the results for pure aluminum material.

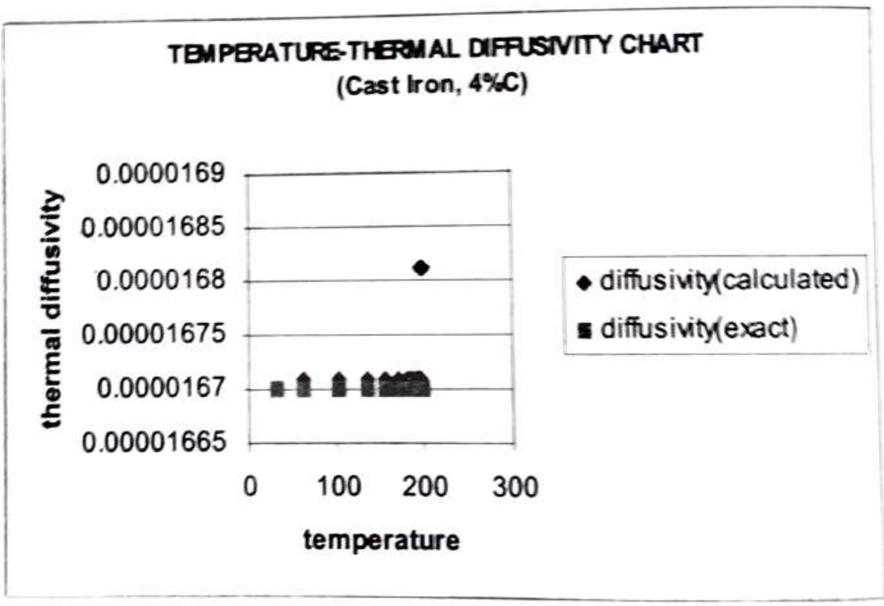

Figure 2.a. Plot of the results for cast iron (4%C) material.

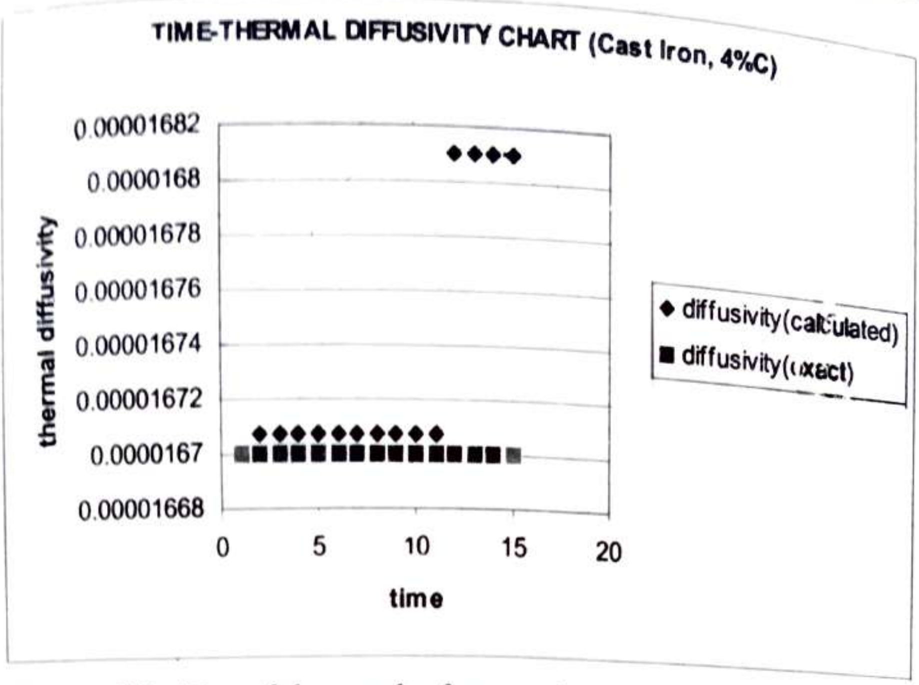

***Figure 2.b.** Plot of the results for cast iron (4%C) material.*

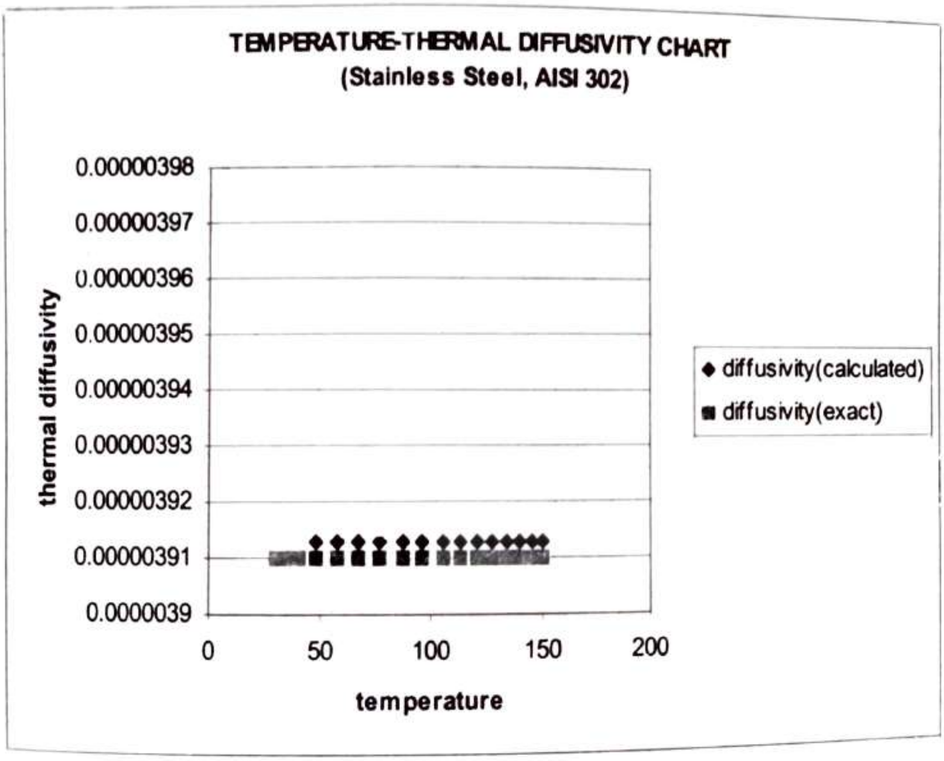

***Figure 3.a.** Plot of the results for stainless steel (AISI 302) material.*

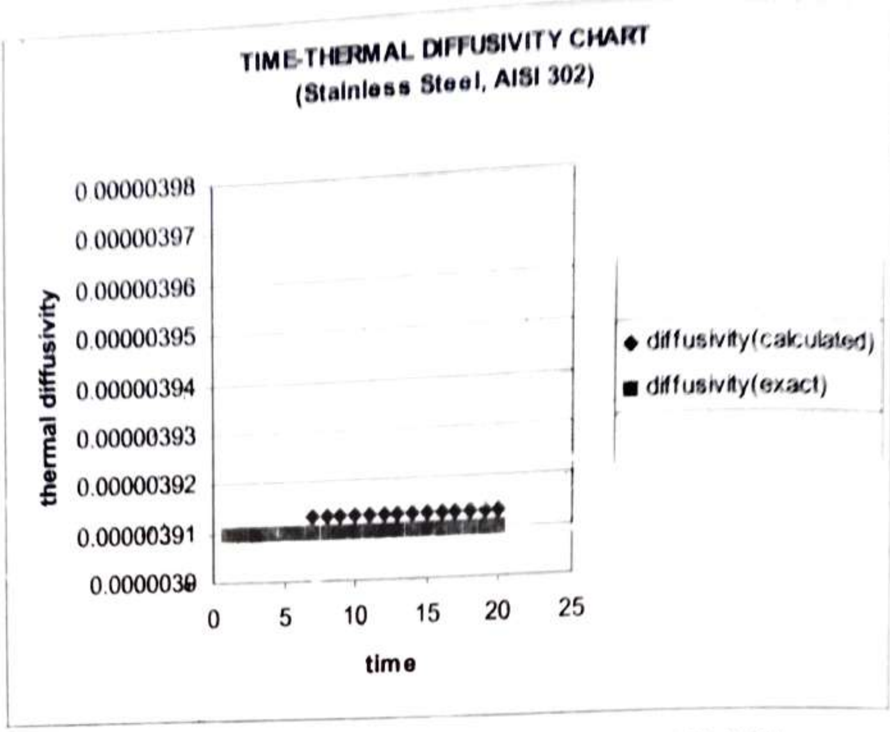

Figure 3.b. *Plot of the results for stainless steel (AISI 302) material.*

The above results clearly indicate that the numerical calculation accurately solved the values of the thermal diffusivity at temperatures not near the steady state temperature, i.e. in this case 200 °C. It does not yield any result after 10,000 iterations at temperatures near the initial temperature of the sphere, i.e., in this case 30 °C, for materials of relatively not very high thermal diffusivity.

Conclusion and Recommendation

The computer program generated can determine the accurate value of the thermal diffusivity given the temperature reading and the elapsed time. It is recommended that experiments be conducted to validate the assumptions used in the derivation of the solution to the general heat conduction partial differential equation. The experimental set-up is such that it will automatically get the reading of the center temperature of the sphere at a particular elapsed time. The reading will then be the input to the computer program in the calculation of the thermal diffusivity.

References

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