HARDIN'S THEORY ON POPULATIONS: TRAGEDY OF THE COMMONS

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INTRODUCTION

In 1968, Garret Hardin, then professor of biology at the University of California, published a paper in Science entitled "The Tragedy of the Commons." The main focus of Hardin's article is on the population problem which he classifies as a member of a class of human problems called the "no technical solution problems," where by a technical solution, it is meant as "one that requires a change only in the techniques of the natural sciences, demanding little or nothing in the way of change in human values or ideas of morality" (Hardin, 1968). In agreement with the Malthusian theory that population tends to grow exponentially, resulting in a decrease in the "per capita share of the world's goods" (where the world is finite), Hardin claims that a finite world, such as ours, "can support only a finite population." Hardin further says that when this ultimate limit of the population is reached, *Bentham's* goal of "the greatest good for the greatest number" cannot be realized. To have the "greatest good for the greatest number" is to have the optimum population less than the maximum.

Hardin's underlying assertion is that "free access to common resource is a recipe for environmental disaster and is, therefore, bound to jeopardize the future well-being of humanity" (Reader, 1988). Hardin illustrates his contention with a situation where the finite world is compared to a finite "pasture open to all." As a rational individual, each herdsman, using the commons, will try to maximize hi return by herding as many cattle as he can. This situation is all right as long as the total cattle population is below the carrying capacity; where the "carrying capacity of a territory is defined as the maximum number of animals that can be supported year after year without damage to the environment" (Hardin, 1988). Tragedy, however, seems to be the end result once the total cattle population reaches the carrying capacity. Hardin reasons out that each man will tend to think in terms of the sole benefits he can derive by adding one more cow to his herd. The "benefits of adding an extra animal to his herd accrued to him alone, the cost of its grazing would be spread among all the herdsmen" (Reader, 1988). Based on this greater "positive utility," a herdsman keeps on adding an extra animal to his herd. All herdsmen think and do the same. "Therein is the tragedy," concludes Hardin. "Each man is locked into a system that compels him to increase his herd without limit – in a world that is limited. Ruin is the destination toward which all men rush, each pursuing his own interest in a society that believes in the freedom of the commons. Freedom in a commons brings ruin to all."

This paper aims to give mathematical models of this tragedy. The stated variables in this study are $C = C(t)$, the cattle population and $H = H(t)$, the herdsmen population. We will consider two situations: the first being cattle and herdsmen population that are both below their respective carrying capacities. We will then demonstrate that *Bentham's* goal of "the greatest good for the greatest number" is not possible. We will show that the optimum population is less than the maximum. The second situation considers the case when the cattle population has reached the carrying capacity and that the herdsmen population is at carrying capacity K_o. We will then demonstrate that in this case, "freedom in a commons brings ruin to all."

MATHEMATICAL MODELS

Basic Assumptions in the Models

We would like to consider a situation in a common grazing land, where the herdsmen population, H, is made up of the same generation that is more or less stable. There are no younger generation herdsmen or if there are, they are only to succeed to the family position as unit herdsmen. The only change in the herdsmen population might be due to immigration into the commons and emigration out of the commons of herdsmen.

For the cattle population, the major cause of increase is through acquisition from external sources. Increase due to birth is assumed to be zero, the commons serving only to fatten cattle and not to nurse the newly born. The major loss is through sales and butchery summed up together as *utilization*. Natural death in the cattle population is also assumed to be zero.

The grassland on which the cattle graze might be assumed to be fairly stable, i.e., if the cattle population is below carrying capacity, the rate of depletion by grazing and mortality equals the rate of replenishment by growth and reproduction. The grazing land's natural carrying capacity of K cows can be thought of as the total number of optimally large and healthy cows that can possibly be raised without the grassland deteriorating. Each herdsmen's fair share of the commons would, therefore, entitle him to herd K/H number of equally large cows.

The carrying capacity of the grassland might also be thought of as the maximum amount of grass biomass that can support K cattle. A given amount of grass biomass could support only so many cows. Increasing the population beyond the carrying capacity of the land would, therefore, reduce the grass available through overgrazing and consequently, reduce the carrying capacity of the land, which in turn reduces the biomass (weight) of each cow in the cattle population. It is therefore, a case of either increasing the number beyond the carrying capacity and reducing the per capita biomass or keeping within the carrying capacity and maintaining optimal biomass.

MODEL FORMULATION: (Populations below carrying capacity)

A. *Cattle Population, C.*

As stated earlier, the increase in the cattle population depends mainly on th rate of addition of new cattle from external sources. We may assume that it is pro portional to the number of herdsmen, H. Also, since it is assumed that the herdsme are rational individuals, an awareness of the carrying capacity of the commons ma limit the rate at which each herdsman is adding extra animals to his herd; hence carrying capacity multiplier (K – C)/K might limit the addition rate (AR). Thu:

$$AR = r_1 H \frac{K - C}{C}, \qquad (1$$

where r_1 (cattle herdsman^{-1} time^{-1}) is the rate in which each herdsman is addin extra animals to his herd.

The main loss of cattle is due principally to sales and butchery, herein combine as Utilization Rate (UR). We may assume that this would be proportional to bot cattle and herdsmen population. Thus,

$$UR = r_2 HC \qquad (2$$

where r_2 (herdsman^{-1} time^{-1}) is the rate of cattle utilization per herdsman.

Thus, using the mass balance equation, the rate of change of the cattle popula tion is given by:

$$\frac{dC}{dt} = r_1 H \frac{K - C}{C} - r_2 HC \qquad (3$$

B. *Herdsman Population, H.*

We will assume that the rate of immigration, IR, into the commons of herds men would be dependent on the proportion of unexploited resource. The propor tion of unexploited resource can be represented by (K – C)/K and $(K_o - H)/K_o$ which express the cattle and human spaces available relative to their respective carrying capacities. Thus,

$$IR = r_3 \frac{K - C}{K} \cdot \frac{K_o - H}{K_o} \qquad (4)$$

where r_3 (herdsmen time^{-1}) is the average rate of immigration of herdsmen into the commons.

The Rate of Emigration (ER) is assumed proportional to both H and C, since a more crowded commons would enhance discontent and, therefore, more emigrations. Hence,

$$ER = r_4 HC \tag{5}$$

where r_4 (cattle $^{-1}$ time $^{-1}$) is the rate of herdsmen emigration per cattle. Combining equations (4) and (5) and using the mass balance equation once more, we obtain the rate of change of the herdsmen population.

$$\frac{dH}{dt} = r_3 \frac{K-C}{K} \cdot \frac{K_o - H}{K_o} - r_4 HC \tag{6}$$

Thus, the dynamical system representing the cattle-herdsmen populations in the commons is given by the pair of differential equations:

$$\frac{dC}{dt} = r_1 H \frac{K-C}{C} - r_2 HC \tag{3}$$

(Model I)

$$\frac{dH}{dt} = r_3 \frac{K-C}{K} \circ \frac{K_o - H}{K_o} - r_4 HC \tag{6}$$

with initial conditions:

$C(O) = C_o > O$, $H(O) = H_o > O$, and where $C_o < K$ and $H_o < K_o$, that is, both initial cattle and herdsman populations are below their respective carrying capacities. The above pair of coupled differential equations will henceforth be referred to in this paper as Model I.

Model Formulation: (Population at or above carrying capacity)

We now consider the situation where the cattle population is at or above the carrying capacity and that the herdsman population is at its carrying capacity K_o, i.e., the social organization has stabilized. This means that $dH/dt = 0$ for all time $t \geqslant O$, hence we only consider the dynamics of the cattle population.

When a population exceeds the carrying capacity of its environment, two important consequences should be noted. Firstly, "the environment is rapidly degraded; as a result, carrying capacity is reduced in subsequent years. Uncontrolled, the population continues to grow larger (for a while) as the carrying capacity grows smaller" (Hardin, 1986). Furthermore, "overexploited edible plants are replaced by weeds previously rejected by the exploiting herbivores. Soil that has been laid bare is eroded away; this reduces local productivity" (Hardin, 1986). Thus, overgrazing results in rapid degradation of and reduction in the productivity of the grassland.

Secondly, evidences suggest that as population density increases, overcrowding stresses the individual members of a population. Moreover, a population under stress need not behave according to accepted unstressed norms. Adult members of a stressed population may, in fact, exhibit behavioral as well as physiological aberrations. As discussed in the book *Ecology and Field Biology* by R.L. Smith, a crowded population of mice held in the laboratory resulted in the suppression of somatic growth and curtailment of reproductive functions in both sexes. Also, rabbits held in crowded living spaces suffered from some debilitating effects (Smith, 1980, p. 495). Thus, an overcrowded population living in a deteriorating environment may exhibit behaviors vastly different from what is expected. Overcrowding, in general, has therefore negative effects on the population. Consequently, equation (3), which is the dynamical representation of the cattle population under normal unstressed conditions, may no longer model the population behavior in an overcrowded environment. There is, therefore, a need to reformulate the dynamical behavior of the cattle population under overcrowding and deteriorating environmental conditions.

It is important at this stage in our modelling exercise to be clarified of the concept of carrying capacity in a deteriorating environment. Following the discussion in Edwards and Fowle (1985), "we may regard carrying capacity as represented by the maximum number of animals of a given species and quality that can, in a given ecosystem, survive through the least favorable environmental conditions occurring within a stated time interval." Thus, the "carrying capacity is not a stable property of a unit environment, but the expression of the interaction of the organisms concerned and their environment." With the above clarification of carrying capacity in a deteriorating environment, the constant K and K_0 in equations (3) and (6) respectively, which we originally defined as the carrying capacity, will now be re-interpreted as the steady state of the population.

Let $B = B(t)$ denote "the maximum population which the environment can support;" that is, the environment can provide all necessary requirements for the maintenance of B individuals, but it will not support $B + 1$ individuals" (Hallam, 1986); that is, B is the carrying capacity of the environment. In general, $B \leqslant K$. But "for species that have evolved in a manner which allows the population to exploit the full potential of the (non-deteriorating) environment, one would expect to have $B = K$" (Hallam, 1986). Following further the development in Hallam (1986), a deteriorating environment can now be modelled by the expression $B = B(t)$ with the properties $B(t) > O$ for all t in R_+ and $B(t) \dashrightarrow O$ as $t \dashrightarrow OO$.

We now consider the per capita increase of the cattle population. With all herdsmen adding extra animals to theird herd, we can write:

$$\begin{pmatrix}\text{Per capita}\\ \text{rate of}\\ \text{increase}\end{pmatrix} = \begin{pmatrix}\text{Cows added}\\ \text{by } K_0\\ \text{herdsmen}\end{pmatrix} - \begin{pmatrix}\text{Utilization}\\ \text{rate}\end{pmatrix} - \begin{pmatrix}\text{Overcrowding}\\ \text{effect}\end{pmatrix} \qquad (7)$$

We now assume that each herdsman adds extra animals to his herd at a constant rate of r_5. Hence,

$$\begin{pmatrix}\text{Cows added by } K_o \\ \text{herdsmen}\end{pmatrix} = r_5 K_o \qquad (8)$$

where r_5(herdsman^{-1} time^{-1}) is the rate of cattle increase per herdsman.

Let us also assume that Utilization Rate is proportional to K_o and C; hence the per capita cattle utilization rate is given by:

$$UR = r_6 K_o C \qquad (9)$$

where r_6 (herdsman^{-1} cows^{-1} time^{-1}) is the per capita utilization rate per herdsman.

We now finally assume that

$$\begin{pmatrix}\text{Overcrowding} \\ \text{effect}\end{pmatrix} = r_7 \frac{C}{B}, \qquad (10)$$

where r_7 (time^{-1}) is a measure of the population's response to environmental stress (Hallam, 1986). Putting equations (B) – (10) in (7), we obtain

$$\frac{1}{C} \circ \frac{dC}{dt} = r_5 K_o - r_6 K_o C - r_7 \frac{C}{B}, \text{ or}$$

$$\frac{dC}{dt} = C[r_5 K_o - r_6 K_o C - r_7 \frac{C}{B}] \qquad \text{(Model II)} \quad -(11)$$

Equation (11) is the dynamics of the cattle population with initial condition $C(O) = C_o \geq K$. Equation (11) will henceforth be called Model II.

QUALITATIVE ANALYSIS OF THE MODELS

We first consider Model I, given by the system of differential equations:

$$\frac{dC}{dt} = r_1 H \frac{K - C}{C} - r_2 HC \qquad \begin{matrix}(3) \\ \text{(Model I)}\end{matrix}$$

$$\frac{dH}{dt} = r_3 \frac{K - C}{K} \cdot \frac{K_o - H}{K_o} - r_{4H}C \qquad (6)$$

with initial conditions $C(O) = C_o \quad O, H(O) = H_o \quad O.$

Equations (5) and (6) can be written in the form

$$\frac{dC}{dt} = r_1 h - [\ \frac{r_1 H}{K} + r_2 H\ ]\ C \tag{12}$$

$$\frac{dH}{dt} = r3\ [\frac{K - C}{K}] - [\ \frac{r_3 K - r_3 C}{KK_o} + r_4 C]\ H \tag{13}$$

We see from equations (12) and (13) that equations (5) and (6) are first order linear equations in their respective variables.

Integral Representations for Model I.

It is convenient to consider the integral representations of the state variables C and H. From equation (12) let

$$g(t) = \frac{r_1 H(t)}{K} + r_2 H(t).$$

We can now write

$$c(t) = e^{-\int_o^t g(s)\, ds} \left[C_o + \int_o^t r_1 H(s)\, e^{\int_{o\, c(t)}^{s} g(u) du}\, ds\right] \tag{14}$$

Also in Equation (13), let

$$h(t) = \frac{r_3 [K - C(t)\ ]}{KK_o} + r_4 C(t).$$

Then:

$$H(t) = e^{-\int_0^t h(s)ds} \left[H_o + \int_0^t r_3\left[\frac{K - C(s)}{K}\right] e^{\int_0^s h(u)du} (ds)\right] \quad (15)$$

Equations (14) and (15) are the integral representations of C and H, respectively. Note that if $C_o > O$ and $H_o > O$ then $C(t) > O$ and $H(t) > O$ for all $t > O$. From biological considerations, these observations are quite important.

We now make the following assertions about Model I, the proof of which is found in the appendix:

ASSERTION I: The ultimate number C* of cattle raised in the commons given

$$C^* = \frac{r_1 K}{r_1 + r_2 K} \quad (16)$$

and the ultimate number H* of herdsmen in the commons is given by:

$$H^* = \frac{r_3 K_o [K - C^*]}{r_3 [K - C^*) + r_4 K C^*} \quad (17)$$

Formally, we write

$$\lim_{t \to \infty} c(t) = C^* \text{ and } \lim_{t \to \infty} H(t) = H^*$$

where C* and H* are given by equations (16) and (17), respectively.

Integral Representation for Model II.

Equation (11) is a modified logistic equation, which is the first order of the Bernoulli type. It has integral representation:

$$C_o e^{\int_0^t r_5 K_o ds}$$

$$c(t) = \frac{\qquad}{1 + C_0 \int_0^t \left[r_6 K_0 + \frac{r_7}{B(s)} \right] e^{\int_0^s r_5 K_0 du} \, ds} \quad (18)$$

We also observe in the integral representation (18) that if $C_0 > O$, then $C(t) > O$ for all $t > O$.

We now make the following assertion about Model II, the proof of which is likewise found in the appendix.

ASSERTION II: A deteriorating environment, $\lim_{t \to O} B(t) = O$, *assures the extinction of the cattle population. Formally, we write: If* $\lim_{t \to O} B(t) = O$, *then* $\lim_{t \to O} C(t) = O$

Thus, overgrazing in the commons ultimately ruins everyone!

Discussion and Computer Simulation

One of the objectives of this study is to illustrate that *Bentham's* goal of "th greatest good for the greatest number" cannot be realized. To show this, it is instruc tive to examine in closer detail the ultimate population levels C^* and H^*. We ca rewrite equation (16) in the form

$$C^* = \frac{K}{1 + \frac{r_2}{r_1} K} \quad (1$$

Hence, it is clear from equation (19) that $C^* < K$, that is, the ultimate number c healthy large cows in the commons is less than the carrying capacity of the con mons.

We can likewise rewrite equation (17) in the form

$$H^* = \frac{K_0}{1 + \frac{r_4 K_0 K C^*}{r_3 (K - C^*)}} \quad (20$$

Since $C^* < K$, we see immediately that $H^* < K_0$. Thus, the ultimate number c both cattle and herdsmen in the commons is less than the optimum. *Bentham* goal is, therefore, not attainable!

It should be noted that the ultimate number of cattle raised in the common is independent of the number of herdsmen. Rather, it depends on the cattle-carrying capacity of the commons, on the rate r_1 at which herdsmen are adding animals to their herd, and on the utilization rate r_2. To maintain a cattle population that is near the carrying capacity K, individual herdsmen should keep on adding cattle to their herd and should minimize the utilization rate r_2. We should be reminded, however, that this practice is viable as long as the cattle population is below the carrying capacity K.

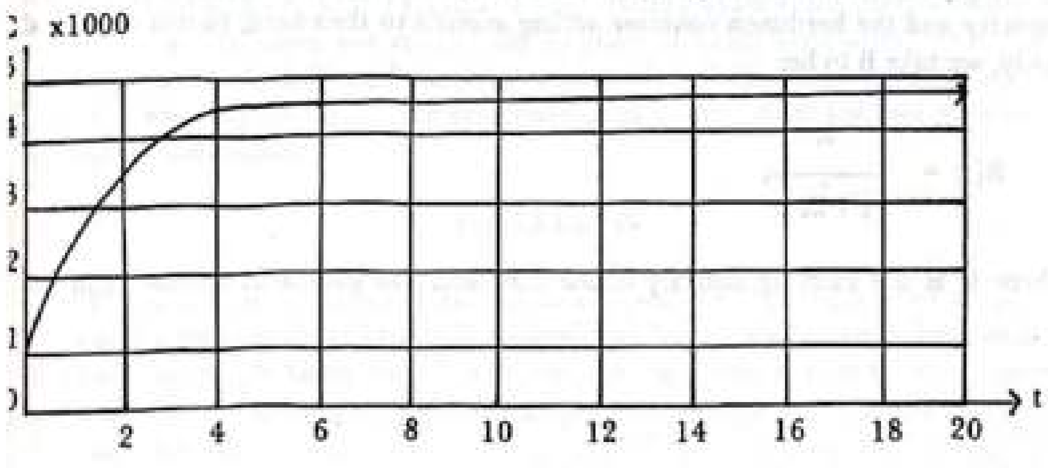
In equation (17), we also note that if C^* is close to K, then H^* is small, that is, fewer herdsmen, will ultimately remain in the commons. This may be explained by the fact that $1 - C^*/K$ represents the unutilized portion of the commons. Hence, if C^* is close to K, then the unexploited portion of the commons is relatively small, discouraging immigration to but encouraging emigration from the commons.

Our assertion in Model II demonstrates Hardin's conclusion that "freedom in a commons brings ruin to all." Disaster is the end-result when the carrying capacity of the environment is systematically exceeded. A conservative approach towards preserving the commons is therefore "to stay well below the best estimate of the carrying capacity" (Hardin, 1986).

Numerical Studies

To illustrate the asymptotic behavior of the cattle and herdsman population, a numerical study was performed using the Turbo Pascal Numerical Methods Toolbox. Specifically, the fourth order Runge-Kutta method for solving initial value problems of coupled first-order ordinary differential equations was used for Model I, while the fourth order Runge-Kutta method for solving initial value problem for a first-order ordinary differential equation was used for Model II.

Figure 1 shows the cattle population of Model I. It is assumed that the commons has a cattle carrying capacity of 5000 cattle and a herdsman carrying capacity of 200 herdsmen. Note that after 20 months, the total cattle production



igure 1. Cattle population component of Model I. Parameter Values used: $r_1 = 40$, $r_2 = 4.0E - 4$. Initial conditions: $C_o = 1000$, $H_o = 50$.

has increased from an initial herd of 1000 cattle to about 4760. The cattle productio is monotonically increasing towards its ultimate population level C* which is belo the carrying capacity of the grassland.

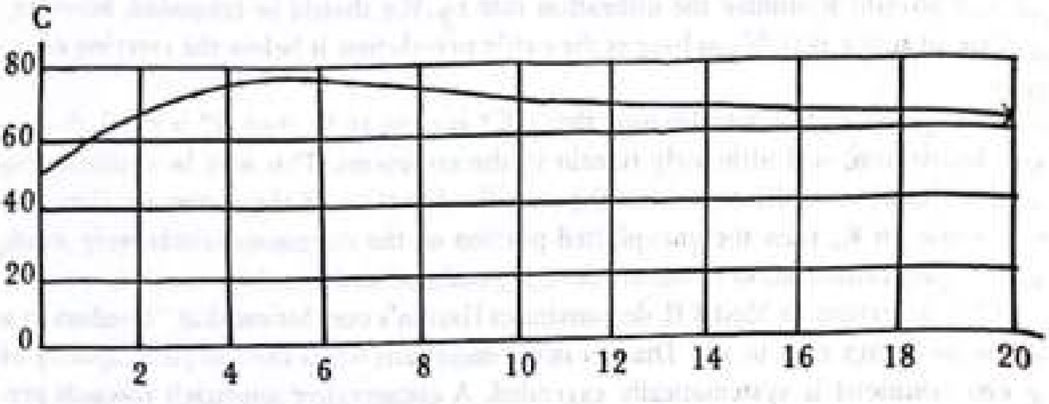


Figure 2. Herdsmen population component of Model I. Parameter values: r_3 = ; r_4 = 33.OE – 7. Initial conditions: C_o = 1000, H_o = 50.

Figure 2 shows the herdsmen population component of Model I. Note th: initially, when the cattle population is still very much below the carrying capacit the number of herdsmen in the commons is increasing. However, as the catt population approaches its carrying capacity, the herdsman population starts t decline. In this numerical study, the herdsman population reaches a peak of abou 73 herdsmen from an initial herdsman population of 50, on or about the 6th month Note that about this time, the cattle population has increased rapidly and is no only gradually increasing to its ultimate population level C*. Apparently, emigratio of herdsmen becomes important as cattle population increases. Hence, we again se in this simulation study that Bentham's goal cannot be achieved. For if we optimiz the cattle population some herdsmen will leave the commons.

Figure 3 illustrates Model II, the case where the cattle population is at carryin capacity and the herdsmen continue adding animals to their herd. In this numerica study, we take B to be:

$$B(t) = \frac{K}{1 + kt},$$

where K is the carrying capacity of the grassland. The parameter K(time^{-1}) in th limited. Moreover, the commons as conceived by Hardin is an open space where ea man is free to do as he pleases. There is no social order that would "ensure that individual could pursue his own interest to the detriment of others," (Read 1988). From the management point of view, it would be prudent then that regu tions or corrective measures that would ensure that the commons would remain a viable source of income not only for the present, but for generations to come, instituted. As to what corrective measures should be formulated is the subject of future investigation.

List of Parameters

Parameter	Unit	Interpretation
r_1	cattle herdsman^{-1} time^{-1}	Rate of cattle addition per herd man
r_2	herdsman^{-1} time^{-1}	Rate of utilization per herdsm
r_3	herdsmen time^{-1}	Average rate of immigration herdsmen into the commons
r_4	cattle^{-1} time^{-1}	Rate of herdsman immigration
r_5	herdsmen^{-1} time^{-1}	Rate of cattle increase
r_6	herdsmen^{-1} cows^{-1} time^{-1}	Per capita utilization rate p herdsman
r_7	time^{-1}	Measure of the population re ponse to environmental stre
K	cattle	Cattle carrying capacity
K_o	herdsmen	Herdsman carrying capacity

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Appendix

Proof of Assertions

Proof of Assertion I:

We rewrite equation (14) in the form

$$c(t) = \frac{C_o + \int_o^t r_1 H(s) e^{\int_o^s g(u)\,du} \, ds}{e^{\int_o^t g(s)ds}} \qquad (21)$$

where $g(t) = \dfrac{r_1 H(t)}{K} + r_2 H(t)$.

We also rewrite equation (15) in the form

$$H(t) = \frac{H_o + \int_o^t r_3 \left[\frac{K - C(s)}{K} \right] e^{\int_o^s h(u)du} \, ds}{e^{\int_o^t h(s)ds}} \qquad (22)$$

$$\text{where } h(t) = \frac{r_3[K - C(t)]}{K_o K} + r_4 C(t)$$

An application of L'Hopital's Rule to (21) and (22) yields the results in (16) (17).

Proof of Assertion II:

Let $C(t,t_o,C_o)$ be the cattle population at time t with initial cattle popula[tion] C_o at time $t = 0$, given by equation (18). Let $\varepsilon > 0$ be given but arbitrary. Si[nce] $\lim_{t \to 0} B(t) = 0$, then there exists $t^* > 0$ such that

$$B(t) < \varepsilon \quad \text{for all } t > t^*.$$

Let $C_1 = C(t^*, t_oC_o)$. Then by uniqueness of solution we have

$C(t,t_oC_o) = C(t,t^*,C_1)$ and:

$$C(t,t^*,C_1) = \frac{C_1 e^{\int_{t^*}^{t} r_5 K_o ds}}{1 + C_1 \int_{t^*}^{t} [r_6 K_o + \frac{r_7}{B(s)}] e^{-\int_{t^*}^{s} r_5 K_o du} ds}$$

We can then write:

$$C(t,t^*,C_1) \quad \frac{\displaystyle\int_{t^*}^{t} r_5K_o\,ds}{\displaystyle\int_{t^*}^{t} \left[r_6K_o + \frac{r_7}{B(s)}\right] e^{\displaystyle r_7\int_{t^*}^{s} r_5K_o\,du}\,ds}$$

$$\frac{1}{\displaystyle\int_{t^*}^{t} \left[r_6K_o + \frac{r_7}{\mathcal{E}}\right] e^{\displaystyle -\int_{s}^{t} r_5K_o\,du}\,ds}$$

$$= \frac{\mathcal{E}\,r_5K_o}{(\mathcal{E}\,r_6K_k + r_1)\,[1 - e^{-r_5K_o(t-t^*)}]}$$

Thus,

$$\lim_{t} \; C(t,t^*,C_1) \quad \frac{\mathcal{E}\,r_5K_o}{\mathcal{E}\,r_6K_o + r_7}$$
$$\quad\; 0$$

Since $\mathcal{E} > 0$ is arbitrary, we must have

$$\lim_{t \to 0} C(t,t_o,C_o) = \lim_{t \to 0} C(t,t^*,C_1) = 0.$$