Characterization of Semi-continuity in the Product Space

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Abstract

In this short note we revisit the concepts of semi-open set and semi-continuity and give some properties of semi-open sets in the Cartesian product with the Tychonoff topology. Further, we characterize semi-continuous functions from a topological space into the product space. The result we obtain runs parallel to the one we have for continuous functions in the product space. Other results involving semi-continuous functions in the product space will also be given.

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1 Introduction

N. Levine introduced the concepts such as semi-open set and semi-continuity in topological spaces [3]. The class of all semi-open sets in a topological

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space includes all open sets. Although an arbitrary union of semi-open sets is semi-open, the class does not always form a topology on the underlying set.

On the other hand, the condition for semi-continuity is strictly weaker than the condition for continuity of a function. However, even for functions into the space R of real numbers with the standard topology, semi-continuity is not generally preserved under algebraic sum, and product of functions.

It is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous where p_α is the αth coordinate projection map. In this paper we give a necessary and sufficient condition for function f to be semi-continuous.

2 Definitions and Known Results

Definition 2.1 Let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let τ_α be the topology on Y_α. The **Tychonoff topology** on $\Pi_{\alpha\in\mathcal{A}} Y_\alpha$ is the topology generated by a subbase consisting of all sets $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha)$, where $p_\alpha : \Pi_{\alpha\in\mathcal{A}} Y_\alpha \to Y_\alpha$ is defined by $p_\alpha((y_\beta)) = y_\alpha$, U_α ranges over all members of τ_α, and α ranges over all elements of $\mathcal{A}$.

We remark that for each open set U_α of Y_α,

$$\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \prod_{\beta\neq\alpha} Y_\beta \ .$$

Hence a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, ..., B_{\alpha_k}\rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, ..., k\}$.

The proofs of the following results can be found in [3].

Theorem 2.2 *Let $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. The projection map $p_\alpha : \Pi_{\alpha\in\mathcal{A}} Y_\alpha \to Y_\alpha$, defined by $p_\alpha((y_\beta)) = y_\alpha$ for each $\alpha \in \mathcal{A}$, is a continuous open surjection.*

Theorem 2.3 *Let $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces and $A_\alpha \subset Y_\alpha$ for each $\alpha \in \mathcal{A}$. Then, in $\Pi_{\alpha\in\mathcal{A}} Y_\alpha$ with the Tychonoff topology,*

$$\overline{\prod_{\alpha\in\mathcal{A}} A_\alpha} = \prod_{\alpha\in\mathcal{A}} \overline{A_\alpha}\ ,$$

where $\overline{A_\alpha}$ is the closure of A_α.

Theorem 2.4 *Let X and Y be topological spaces. A function $f : X \to Y$ is continuous on X if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every set $A \subseteq X$.*

Definition 2.5 Let X be a topological space. A set $O \subseteq X$ is semi-open in X if there exists an open set G in X such that $G \subseteq O \subseteq \overline{G}$.

Definition 2.6 Let X and Y be topological spaces. A function $f : X \to Y$ is semi-continuous on X if the inverse image of every open set in Y is semi-open in X, i.e.,

$$G \text{ open in } Y \Rightarrow f^{-1}(G) \text{ is semi-open in } X.$$

In simple terms, we call an element of a basis of a topological space a *basic open set*. Similarly, an element of a subbase will be referred to as a *subbasic open set*. In [1], the authors proved the following result.

Theorem 2.7 *Let X and Y be topological spaces. A function $f : X \to Y$ is semi-continuous on X if and only if the inverse image of every basic (subbasic) open set in Y is semi-open in X.*

3 Results

Throughout this section, the Cartesian product $Y = \Pi_{\alpha \in \mathcal{A}} Y_\alpha$ carries the Tychonoff topology. This topological space is referred to as the product space.

Lemma 3.1 *If O is a non-empty semi-open set in the product space Y, then $p_\alpha(O) = Y_\alpha$ for all but at most finitely many α and $p_\alpha(O)$ is semi-open for every $\alpha \in \mathcal{A}$.*

Proof: There exists a non-empty open set G in Y such that $G \subseteq O \subseteq \overline{G}$. Thus,

$$p_\alpha(G) \subseteq p_\alpha(O) \subseteq p_\alpha(\overline{G})$$

for every $\alpha \in \mathcal{A}$. By Theorem 2.4, we have

$$p_\alpha(G) \subseteq p_\alpha(O) \subseteq \overline{p_\alpha(G)}$$

for every $\alpha \in \mathcal{A}$. Since G is a non-empty open set, G contains some basic open set $B = \langle B_{\alpha_1}, B_{\alpha_2}, ..., B_{\alpha_k} \rangle$. Since $p_\alpha(B) = Y_\alpha$ for all $\alpha \notin K = \{\alpha_1, \alpha_2, ..., \alpha_k\}$ and $p_\alpha(B) \subset p_\alpha(G)$, it follows that $p_\alpha(G) = Y_\alpha$ for all but at most finitely many α. Hence $p_\alpha(O) = Y_\alpha$ for all but at most finitely many α. Next, fix $\alpha \in \mathcal{A}$. Then either $p_\alpha(O) = Y_\alpha$ or $p_\alpha(O) \neq Y_\alpha$. If $p_\alpha(O) = Y_\alpha$, then $p_\alpha(O)$ is semi-open. So suppose that $p_\alpha(O) = O_\alpha \neq Y_\alpha$ and let $p_\alpha(G) = G_\alpha$. Then

$$G_\alpha \subseteq O_\alpha \subseteq \overline{G_\alpha}\,.$$

Since the projection map p_α is open, $p_\alpha(G)$ is open in Y_α. Therefore $p_\alpha(O)$ is semi-open for all $\alpha \in \mathcal{A}$. $\square$

Remark 3.2 *The converse of Lemma* 3.1 *is not true.*

To see this, consider $Y_1 = \{1,2,3\}$ with the topology $\tau_1 = \{Y_1, \emptyset, \{2\}, \{1,2\}, \{2,3\}\}$ and $Y_2 = \{a,b,c,d\}$ with topology $\tau_2 = \{Y_2, \emptyset, \{a\}, \{c\}, \{a,c\}\}$. Let $O = \{(1,a),(2,d),(3,b)\}$. Then the family $\mathcal{B}$ consisting of the sets $Y_1 \times Y_2$, $\emptyset$, $Y_1 \times \{a\}$, $Y_1 \times \{c\}$, $Y_1 \times \{a,c\}$, $\{2\} \times Y_2$, $\{1,2\} \times Y_2$, $\{2,3\} \times Y_2$, $\{(1,a),(2,a)\}$, $\{(1,c),(2,c)\}$, $\{(2,a)\}$, $\{(2,c)\}$, $\{(2,a),(2,c)\}$, $\{(2,a),(3,a)\}$, $\{(2,c),(3,c)\}$, and $\{(2,a),(2,c),(3,a),(3,c)\}$ is a basis for the Tychonoff topology on $Y_1 \times Y_2$. Since $p_1(O) = \{1,2,3\} = Y_1$ and Y_1 is open (hence, semi-open), $p_1(O)$ is semi-open. Now, $p_2(O) = \{a,b,d\}$. Clearly, $\{a\} \subseteq p_2(O)$. Since $\overline{\{a\}} = \{a,b,d\}$, we find that $\{a\}$ is an open set in Y_2 satisfying $\{a\} \subseteq p_2(O) = \overline{\{a\}}$. This implies that $p_2(O)$ is also semi-open. However, O is not semi-open because O does not contain a non-empty basic open set in $Y_1 \times Y_2$.

Theorem 3.3 *Let $S = \{\alpha_1, \alpha_2, ..., \alpha_k\}$ be a finite subset of $\mathcal{A}$ and $\varnothing \neq O_{\alpha_i} \subseteq Y_{\alpha_i}$ for each $\alpha_i \in S$. Then $\langle O_{\alpha_1}, O_{\alpha_2}, ..., O_{\alpha_k} \rangle$ is semi-open in Y if and only if each O_{α_i} is semi-open in Y_{α_i}.*

Proof: Let $O = \langle O_{\alpha_1}, O_{\alpha_2}, ..., O_{\alpha_k} \rangle$ and suppose each O_{α_i} is a non-empty semi-open set in Y_{α_i}. Then there exists an open set G_{α_i} in Y_{α_i} such that $G_{\alpha_i} \subseteq O_{\alpha_i} \subseteq \overline{G_{\alpha_i}}$. Let $G = \langle G_{\alpha_1}, G_{\alpha_2}, ..., G_{\alpha_k} \rangle$. Then G is open in Y and $\overline{G} = \langle \overline{G_{\alpha_1}}, \overline{G_{\alpha_2}}, ..., \overline{G_{\alpha_k}} \rangle$ by Theorem 2.2. Thus, G is an open set in Y satisfying $G \subseteq O \subseteq \overline{G}$. This shows that O is semi-open in Y.

Conversely, suppose O is a non-empty semi-open set in Y. By Lemma 3.1, $p_{\alpha_i}(O)$ is semi-open in Y_{α_i} for every $i \in \{1, 2, ..., k\}$. It follows that each O_{α_i} is semi-open in Y_{α_i}.

The proof of the theorem is complete. □

We shall now characterize semi-continuous functions from an arbitrary topological space X into the product space Y.

Theorem 3.4 *A function $f : X \to Y$ is semi-continuous on X if and only if each coordinate function $p_\alpha \circ f$ is semi-continuous on X.*

Proof: Suppose f is semi-continuous on X. Let $\alpha \in \mathcal{A}$ and U_α be open in Y_α. Since p_α is continuous, $p_\alpha^{-1}(U_\alpha)$ is open in Y. Hence,

$$f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)$$

is a semi-open set in X. Thus, $p_\alpha \circ f$ is semi-continuois for every $\alpha \in \mathcal{A}$, by Definition 2.6.

Conversely, suppose each coordinate function $p_\alpha \circ f$ is semi-continuous. Let G_α be open in Y_α. Then $\langle G_\alpha \rangle$ is a subbasic open set in Y and

$$(p_\alpha \circ f)^{-1}(G_\alpha) = f^{-1}(p_\alpha^{-1}(G_\alpha) = f^{-1}(\langle G_\alpha \rangle)$$

is a semi-open set in X. Therefore, f is semi-continuous on X, by Theorem 2.7. □

Corollary 3.5 *Let X be a topological space, Y the product space and $f_\alpha : X \to Y_\alpha$ a function for each $\alpha \in \mathcal{A}$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x) \rangle$. Then f is semi-continuous on X if and only if f_α is semi-continuous for each $\alpha \in \mathcal{A}$.*

Proof: For each $\alpha \in \mathcal{A}$ and each $x \in X$, we have

$$(p_\alpha \circ f)(x) = p_\alpha(f(x)) = p_\alpha(\langle f_\alpha(x) \rangle) = f_\alpha(x).$$

Thus $p_\alpha \circ f = f_\alpha$ for every $\alpha \in \mathcal{A}$. The result now follows from Theorem 3.4. □

Theorem 3.6 *Let X and Y be the product spaces of the families of spaces $\{X_\alpha : \alpha \in \mathcal{A}\}$ and $\{Y_\alpha : \alpha \in \mathcal{A}\}$, respectively. For each $\alpha \in \mathcal{A}$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function. If each f_α is semi-continuous, then the function $f : X \to Y$, defined by $f(\langle x_\alpha \rangle) = \langle f_\alpha(x_\alpha) \rangle$, is semi-continuous on X.*

Proof: Let $\langle V_\alpha \rangle$ be a subbasic open set in Y. Then

$$f^{-1}(\langle V_\alpha \rangle) = \langle f_\alpha^{-1}(V_\alpha) \rangle .$$

Since f_α is semi-continuous, $f_\alpha^{-1}(V_\alpha)$ is semi-open in X_α. Hence there exists an open set G_α in X_α such that

$$G_\alpha \subseteq f_\alpha^{-1}(V_\alpha) \subseteq \overline{G_\alpha} .$$

Clearly $\langle G_\alpha \rangle$ is an open set (subbasic open) in X and

$$\langle G_\alpha \rangle \subseteq \langle f_\alpha^{-1}(V_\alpha) \rangle \subseteq \langle \overline{G_\alpha} \rangle .$$

This implies that $\langle f_\alpha^{-1}(V_\alpha) \rangle$ is a semi-open set in Y. Thus, f is semi-continuous on X. □

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