Counting Restricted Functions on a Formation of Points Having Positive Integral Coordinates

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Abstract

In this paper we establish some formulas in counting restricted functions $f_{|S}$ under each of the following conditions:

(i) $f(a) \leq g(a)$, $\forall a \in S$ where g is any non-negative real-valued continuous function.

(ii) $g_1(a) \leq f(a) \leq g_2(a)$, $\forall a \in S$ where g_1 and g_2 are any two non-negative real-valued continuous functions.

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1 Introduction

Consider a mapping f from N_m to N_n where $N_k = \{1, 2, \ldots, k\}$. According to Cantor's proposition on counting function [4], the number of possible function we can form from this mapping is equal to n^m. However, if we restrict our domain to $S_i \subseteq N_m$ with $|S_i| = i$, then there are n^i possible restricted functions we can form. When Ψ is the set of all $f_{|S_i}$ overall $S_i \subseteq N_m$ with $|Si| = i$,

$$\Psi_{i,m} = \left| \bigcup_{S_i \subseteq N_m} \{f_{|S_i}\} \right| = \binom{m}{i} n^i .$$

Thus, with $\Psi_m = \bigcup_{i=0}^{n} \Psi_{i,m}$,

$$|\Psi_m| = \sum_{i=0}^{m} |\Psi_{i,m}| = \sum_{i=0}^{m} \binom{m}{i} n^i = (1+n)^m .$$

This means that the total number of restricted functions $f_{|S}$ overall $S \subseteq N_m$ is equal to $(1+n)^m$.

2 The Condition $f(a) \leq g(a)$

In this section, we will consider a function $f : N_m \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. We are going to count the number of restricted functions $f_{|S}$ such that $f(a) \leq g(a)$, $\forall a \in S \subseteq N_m$ where g is any non-negative real-valued continuous function. Note that an element $a \in S$ can be mapped to any of the natural numbers $1, 2, \ldots, \lceil g(a) \rceil$, where $\lceil g(a) \rceil$ is the greatest integer that is less than or equal to a real number $g(a)$. Hence, if

$S_i = j_1, j_2, \ldots, j_i$, then

$$\left|\{f_{|S_i}\}\right| = \prod_{l=1}^{i} \lceil g(j_l) \rceil$$

with $|\{f_{|S_0}\}| = 1$. Thus, with $\hat{\Psi}_{i,m} = \bigcup_{S_i \subseteq N_m} \{f_{|S_i}\}$,

$$|\hat{\Psi}_{i,m}| = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le m} \prod_{l=1}^{i} \lceil g(j_l) \rceil \ .$$

This result is embodied in the following proposition.

Proposition 2.1 *Let f be a function from N_m to $\mathbb{N}$ such that $f(a) \le g(a)$, $\forall a \in N_m$ where g is any nonnegative real-valued continuous function. Then the number of restricted functions $f_{|S_i}$ overall $S_i \subseteq N_m$ such that $|S_i| = i$ is given by*

$$|\hat{\Psi}_{i,m}| = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le m} \prod_{l=1}^{i} \lceil g(j_l) \rceil \ .$$

where $|\hat{\Psi}_{0,m}| = 1$, $|\hat{\Psi}_{i,m}| = 0$ when $i > m$.

For a quick computation of the first values of $|\hat{\Psi}_{i,m}|$, we have the following recurrence relation.

Proposition 2.2 *The number $|\hat{\Psi}_{i,m}|$ satisfies the following recurrence relation:*

$$|\hat{\Psi}_{i,m+1}| = |\hat{\Psi}_{i,m}| + \lceil g(m+1) \rceil \, |\hat{\Psi}_{i-1,m}|$$

Proof: Note that $|\hat{\Psi}_{i,m+1}|$ counts the number of restricted functions $f_{|S_i}$ overall $S_i \subseteq N_m$. This number can also be counted by considering the following cases:

Case 1. $m+1 \notin S_i$. Counting the desired restricted functions under this case is equivalent to counting restricted functions $f_{|S_i}$ overall $S_i \subseteq N_m$. Hence, there are $|\hat{\Psi}_{i,m}|$ such restricted functions.

Case 2. $m+1 \in S_i$. Counting the desired restricted functions under this case is equivalent to the following sequence of events: i) counting restricted functions $f_{|S_i}$ overall $S_{i-1} \subseteq N_m$, which is equal to

$$\sum_{1 \le j_1 < j_2 < \cdots < j_l \le m} \prod_{l=1}^{i-1} \lceil g(j_l) \rceil .$$

ii) insert $m+1$ to every S_{i-1} and map $m+1$ to any of the natural numbers $1, 2, \ldots, \lceil g(m+1) \rceil$. By Multiplication Principle (MP), the number of such restricted functions $f_{|S_i}$ with $S_i = S_{i-1} \cup \{m+1\}$ is equal to

$$\lceil g(m+1) \rceil \sum_{1 \le j_1 < j_2 < \cdots < j_l \le m} \prod_{l=1}^{i-1} \lceil g(j_l) \rceil = \lceil g(m+1) \rceil \, |\hat{\Psi}_{i-1,m}| .$$

Using Addition Principle (AP), we prove the proposition. □

We are now ready to prove the following proposition.

Proposition 2.3 *Let* $\hat{\Psi}_m = \bigcup_{i=0}^{m} \hat{\Psi}_{i,m}$. *Then*

$$|\hat{\Psi}_m| = \prod_{i=1}^{m} (1 + \lceil g(i) \rceil)$$

with $|\hat{\Psi}_0| = 1$.

Proof: We will prove this by induction on m. For $m = 0$,

$$|\hat{\Psi}_0| \; = \; |\hat{\Psi}_{0,0}| = \left| \bigcup_{S_0 \subseteq N_m} \{f_{|S_0}\} \right| = 1 .$$

For $m = 1$, with the help of Proposition 2.2, we have

$$\begin{aligned} |\hat{\Psi}_1| &= |\hat{\Psi}_{0,1}| + |\hat{\Psi}_{1,1}| \\ &= 1 + 1\lceil g(1)\rceil \\ &= \prod_{i=1}^{1}(1 + \lceil g(i)\rceil)\,. \end{aligned}$$

Suppose it is true for $k > 1$, i.e.,

$$|\hat{\Psi}_k| = \sum_{i=0}^{k} |\hat{\Psi}_{i,k}| = \prod_{i=1}^{k}(1 + \lceil g(i)\rceil)\,.$$

Then, by Proposition 2.2, we have

$$\begin{aligned} |\hat{\Psi}_{k+1}| &= \sum_{i=0}^{k+1} |\hat{\Psi}_{i,k+1}| \\ &= \sum_{i=0}^{k+1} |\hat{\Psi}_{i,k}| + \lceil g(k+1)\rceil \sum_{i=0}^{k+1} |\hat{\Psi}_{i-1,k}| \\ &= \sum_{i=0}^{k} |\hat{\Psi}_{i,k}| + \lceil g(k+1)\rceil \sum_{i=0}^{k} |\hat{\Psi}_{i,k}| \\ &= |\hat{\Psi}_k|(1 + \lceil g(k+1)\rceil)\,. \end{aligned}$$

Using the inductive hypothesis, we obtain

$$|\hat{\Psi}_{k+1}| = \prod_{i=1}^{k+1}(1 + \lceil g(i)\rceil)\,. \qquad \square$$

Example 2.4 *Given f a function on N_5. If $g(a) = 2a^3 + 3$, how many possible restricted functions $f_{|S}$ can be formed overall $S \subseteq N_5$ such that $f(a) \leq g(a)$ for every $a \in N_5$?*

Solution: The values of $\lceil g(a)\rceil$, $a = 1, 2, \ldots, 5$ are shown in the table below:

A	1	2	3	4	5
$\lceil g(a) \rceil$	5	19	57	131	253

Table 1

Using Proposition 2.3, the number of restricted functions $f_{|S}$ that can be formed is

$$\begin{aligned} |\hat{\Psi}_5| &= (6)(20)(58)(132)(254) \\ &= 233,354,880 \, . \end{aligned}$$

Remark 2.5 1. *When $g(a) = n$, $n \in \mathbb{N}$, we have*

a. $|\hat{\Psi}_{i,m}| = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le m} n^i = \binom{m}{i} n^i = |\hat{\Psi}_{i,m}|$

b. $|\hat{\Psi}_m| = \prod_{i=1}^{m} (1+n) = (1+n)^m = |\Psi_m|$

2. *When $g(a) = a$, $a \in \mathbb{N}_m$, we have*

a. $|\hat{\Psi}_{i,m}| = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le m} \prod_{l=1}^{i} \lceil g(j_l) \rceil = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le m} \prod_{l=1}^{i} j_l$

b. $|\hat{\Psi}_m| = \prod_{i=1}^{m} (1 + \lceil g(i) \rceil) = \prod_{i=1}^{m} (1+i) = (1+m)!$.

3 The Condition $g_1(a) \le f(a) \le g_2(a)$

In this section, we are going to count the number of restricted functions $f_{|S} : S \to \mathbb{N}$ such that $g_1(a) \le f(a) \le g_2(a)$, $a \in S \subseteq N_m$ where g_1 and g_2 are any two nonnegative real-valued continuous functions. It can easily be seen that an element a S can be mapped to any of the natural numbers $\lfloor g_1(a)\rfloor, \lfloor g_1(a)\rfloor + 1 \ldots, \lceil g_2(a)\rceil$ where $\lfloor g_1(a)\rfloor$ is the least integer that is greater than or equal to real number $g1(a)$. Hence, each $a \in S_i$ can be paired to $\lceil g_2(a)\rceil - \lfloor g_1(a)\rfloor + 1$ elements of $\mathbb{N}$. Thus, if $S_i = \{j_1, j_2, \ldots, j_i\}$, then

$$|f_{|S_i}| = \prod_{l=1}^{i} \bar{g}(j_l), \bar{g}_i = \lceil g_2(j_l)\rceil - \lfloor g_1(j_l)\rfloor + 1 .$$

Thus, with $\tilde{\Psi}_{i,m} = \bigcup_{S_i \subseteq N_m} \{f_{|S_i}\}$,

$$|\tilde{\Psi}_{i,m}| = \sum_{1\leq j_1<j_2<\cdots<j_i\leq m} \prod_{l=1}^{i} \bar{g}(j_l) .$$

This result is embodied in the following proposition:

Proposition 3.1 *Let f be a function from N_m to $\mathbb{N}$ such that $g_1(a) \leq f(a) \leq g_2(a)$, $a \in S \subseteq N_m$ where g_1 and g_2 are any two nonnegative real-valued continuous functions. Then the number of restricted functions $f|S_i$ overall $S_i \subseteq N_m$ such that $|S_i| = i$ is given by*

$$|\bar{\Psi}_{i,m}| = \sum_{1\leq j_1<j_2<\cdots<j_i\leq m} \prod_{l=1}^{i} \bar{g}(j_l), \tilde{j}_l = \lceil g_2(a)\rceil - \lfloor g_1(a)\rfloor + 1$$

where $|\tilde{\Psi}_{0,m}| = 1$, $|\bar{\Psi}_{i,m}| = 0$ when $i > m$.

The next proposition is a recurrence relation of $|\bar{\Psi}_{i,m}|$, which can be proven using the same argument as in the proof of Proposition 2.2.

Proposition 3.2 *Let $\tilde{\Psi}_m = \bigcup_{i=0}^{m} \tilde{\Psi}_{i,m}$. Then*

$$|\tilde{\Psi}_m| = \prod_{i=1}^{m} (1 + \bar{g}(i))$$

with $|\tilde{\Psi}_0| = 1$.

To illustrate these results, let us consider the following example.

Example 3.3 Let $g_1(x) = \frac{21}{25}x^2$, $g_2(x) = \frac{21}{5}x$, and $f : N_5 \to \mathbb{N}$ such that $g_1(a) \le f(a) \le g_2(a)$. How many possible restricted functions $f_{|S}$ overall $S \subseteq N_5$ can we form? How many of these restricted functions whose domain is S_i, $i = 0, 1, 2, 3, 4, 5$?

Solution: First, let us construct a table of values for $\tilde{g}(a)$:

a	1	2	3	4	5
$g_2(a)$	4.2	8.4	12.6	16.8	21
$g_1(a)$	0.84	3.36	7.56	13.44	21
$\lceil g_2(a) \rceil$	4	8	12	16	21
$\lfloor g_1(a) \rfloor$	1	4	8	14	21
$\tilde{g}(a)$	4	5	5	3	1

Table 2

Now, using Proposition 3.2, the total number of restricted functions $f_{|S}$ overall $S \subseteq N_5$ is given by

$$|\tilde{\Psi}_5| = (1+4)(1+5)(1+5)(1+3)(1+1) = 5(6)(6)(4)(2) = 1440 .$$

On the other hand, using Proposition 3.1, the number of restricted functions $f_{|S}$, $i = 0, 1, 2, \ldots, 5$, can be computed as follows:

$$\begin{aligned}
i = 0,\ |\tilde{\Psi}_{0,5}| &= 1 \\
i = 1,\ |\tilde{\Psi}_{1,5}| &= 4 + 5 + 5 + 3 + 1 = 18 \\
i = 2,\ |\tilde{\Psi}_{2,5}| &= 4(5) + 4(5) + 4(3) + 4(1) + 5(5) + 5(3) + 5(1) + 5(3) \\
&\quad + 5(1) + 3(1) = 124
\end{aligned}$$

$$
\begin{aligned}
i = 3,\ |\tilde{\Psi}_{3,5}| &= 4(5)(5) + 4(5)(3) + 4(5)(1) + 4(5)(3) + 4(5)(1) + 4(3)(1) \\
&\quad +5(5)(3) + 5(5)(1) + 5(3)(1) + 5(3)(1) = 402 \\
i = 4,\ |\tilde{\Psi}_{4,5}| &= 4(5)(5)(3) + 4(5)(5)(1) + 4(5)(3)(1) + 4(5)(3)(1) + 5(5)(3) \\
&= 595 \\
i = 5,\ |\tilde{\Psi}_{5,5}| &= 4(5)(5)(3)(1) = 300\,.
\end{aligned}
$$

Note that

$$\sum_{i=0}^{5} |\widetilde{i,5}| = 1 + 18 + 124 + 402 + 595 + 300 = 1440 = |\tilde{\Psi}_5|\,.$$

Remark 3.4 *When* $g_1(a) = 1$ *and* $g_2(a) = g(a)$, $\forall a \in \mathbb{N}$, $\tilde{g}(a) = \lceil g(a) \rceil$ *and*

(a) $|\tilde{\Psi}_{i,m}| = |\Psi_{i,m}|$

(b) $|\tilde{\Psi}_m| = |\Psi_m|$.

4 Recommendation

The functions that we are counting here are functions of one variable which contain points on the x-y plane with positive integral coordinates. For possible future research, it is also interesting to consider a function whose elements are points with positive integral coordinates on an n-dimensional space where $n \geq 3$. The authors believe that the results obtained in this paper can be extended to a more general case by counting such restricted functions. Moreover, it is also worth considering those restricted functions on an n-dimensional space, which are one-to-one and onto. There are already results in [1] about counting one-to-one and onto functions of one variable. One may try to apply the method used in [1] to count the number of restricted one-to-one and onto functions on an n-dimensional space.

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