On The Primitives of Some Bilinear Henstock-Stieltjes Integrable Functions

Ferdinand P. Jamil
Sergio R. Canoy, Jr.

Abstract

Primitives play a key role in many significant results in Henstock integration. For instance, they are used to characterize absolute integrable functions. The controlled convergence theorem which is the best possible convergence theorem for the Henstock integral involves primitives.

Bilinear Henstock-Stieltjes integral had been the subject of study in previous works of the authors. This paper investigates primitives of bilinear Henstock-Stieltjes integrable functions under certain conditions on the integrands and integerators. It is known that in the case of bilinear Stieltjes integrals, a primitive, up to some extent, is influenced by its integrator. Interestingly, it has been known that unlike in the real and ordinary case, a primitive is not necessarily continuous.

Keywords: regulated function, function of bounded variation, absolutely continuous functions, bilinear Henstock-Stiletjes integral, primitives.

FERDINAND P. JAMIL is a Professor of Mathematics in the Department of Mathematics, Mindanao State University, Marawi City. He holds a Ph.D. in Mathematics from Mindanao State University, Iligan Institute of Technology, Iligan City. **SERGIO R. CANOY, JR.** is a Professor of Mathematics in the College of Science and Mathematics, MSU-IIT, Iligan City.

1 Preliminaries

Let us recall a few concepts mentioned in [1], [3] and [7]. Here we use X, Y and Z to denote real Banach spaces, and $L(X, Y, ; Z)$ to stand for the space of all bounded bilinear transformations $A : X \times Y \rightarrow Z$. By a *bilinear transformation* $A : X \times Y \longrightarrow Z$, it has the property that for all $x, x_1 \in X$, $y, y_1 \in Y$ and $\alpha \in \mathbf{R}$, we have

(*i*) $A(x + x_1, y) = A(x, y) + A(x_1, y)$

(*ii*) $A(x, y + y_1) = A(x, y) + A(x, y_1)$

(*iii*) $A(\alpha x, y) = \alpha A(x, y)$

(*iv*) $A(x, \alpha y) = \alpha A(x, y)$

A is said to be *bounded* whenever there is a positive number M such that

$$\|A(x, y)\| \leq M \cdot \|x\| \cdot \|y\|$$

for all $x \in X$, $y \in Y$. In $L(X, Y, ; Z)$ we define the norm $\|.\|$ by

$$\|A\| = inf\{M : \|A(x, y)\| \leq M\|x\|\|y\| \ for \ \ any \ x \in X \ and \ y \in Y\}.$$

Let $f : [a, b] \rightarrow X$ be any function. We say that f is *regulated* if for every $\xi \in [a, b)$, the right-sided limit $f(\xi+)$ exists and is finite, and for $\xi \in (a, b]$ the left-sided limit $f(\xi-)$ exists and is finite. A regulated function f has a countable number of points of discontinuity.

We say that f is of *bounded variation* on $[a, b]$ whenever

$$Var(f; [a, b]) = sup(P) \sum \|f(t_i) - f(t_{i-1})\| < \infty,$$

where the supremum is taken over all partitions $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$. A function of bounded variation is regulated **[6]**.

Function f is *absolutely continuous* on $[a, b]$ whenever given $\epsilon > 0$, there exists $\delta > 0$ such that if $\{[a_i, b_i]\}$ is a finite or infinite sequence of non-overlapping intervals in $[a, b]$ we have

$$\sum_i |b_i - a_i| < \delta \ \textit{implies} \ \sum_i \|f(b_i) - f(a_i)\| < \epsilon.$$

An absolutely continuous function on $[a, b]$ is continuous on the same interval $[a, b]$. The converse, however, is not true.

We now turn to bilinear Henstock-Stieltjes integral. Given a positve function $\delta(\xi)$ on $[a, b]$, a *δ-fine division* is a set of interval-point pairs $D = \{([a, b]; \xi)\}$, where the $[u, v]$'s form a partition of $[a, b]$ and for every pair $([u, v]; \xi)$ in D, we have

$$\xi \in [u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi)).$$

Any subset of a δ-fine division of $[a, b]$ is a *δ-fine partial division* of $[a, b]$.

Let $A \in L(X, Y; Z)$ and let $f : [a, b] \to X$ and $g : [a, b] \to Y$ be any functions.We say that f is *Henstock-Stieltjes integrable* with respect to A and g on $[a, b]$ if there is a vector J in Z satisfying the following condition: For every $\epsilon > 0$ there exists a positive function $\delta(\xi)$ defined on $[a, b]$ such that for any δ-fine division $D = \{([u, v]); \xi)\}$ of $[a, b]$,

$$\|(D) \sum A(f(\xi), g(v) - g(u)) - J\| < \epsilon.$$

In this, we write

$$J = (HS) \int_a^b A(f, dg).$$

We also include the definition

$$J = (HS)\int_a^a A(f, dg) = 0.$$

"'HS"' stands for Henstock-Stieltjes, and the integral is called *bilinear Henstock Stieltjes integral*. If no confusion arises, we may as well omit "HS" for simplicity.

For the algebraic properties and other related results concerning the integral, the reader may refer to [**1**], [**2**], [**3**] and [**4**].

2 Results

Let $A \in L(X, Y; Z)$, and let $f : [a, b] \to X$ be Henstock-Stieltjes integrable with respect to A and $g : [a, b] \to Y$ on $[a, b]$. If $F : [a, b] \to Z$ is given by

$$F(x) = \int_a^x A(f, dg)$$

for all $x \in [a, b]$, then for any partition $D = \{[u, v]\}$ of $[a, b]$, we have

$$(D)\sum\{F(v) - F(u)\} = \int_a^b A(f, dg).$$

Lemma 2.1 *Let $A \in L(X, Y; Z)$, and let $f : [a, b] \to X$ be Henstock-Stieltjes integrable with respect to A and $g : [a, b] \to Y$ on $[a, b]$. There exists a function $F : [a, b] \to Z$ with the property: Given $\epsilon > 0$, there is a positive function $\delta(\xi)$ on $[a, b]$ such that for any δ-fine division $D = \{([u, v]; \xi)\}$ of $[a, b]$,*

$$\|(D)\sum\{F(v) - F(u) - A(f(\xi), g(v) - g(u))\}\| < \epsilon.$$

A function F that has the property in Lemma 1 is called a *primitive* of f with respect to A and g on $[a, b]$. For convenience, we write $F(u, v)$ for the difference $F(v) - F(u)$. It is worth noting that if F is a primitive, then $F(a,b)$ is the value of the integral.

If $Y = Z = \mathbf{R}$ and A is the scalar product: $A(x,\alpha) = \alpha \cdot x$, and if $g : [a,b] \to Y$ is the identity function $g(x) = x$, then the integral $\int_a^b A(f,dg)$ turns out to be the integral $\int_a^b f$ discussed in **[5]** by Sergio Cao. It is known that if F is a primitive of $\int_a^b f$, then F is always continuous on $[a,b]$**[5]**. This means F is continuous regardless of whether or not f is continuous on $[a,b]$. Therefore, the continuity of a primitive F of f cannot be attributed to the function f. If so, then what must have guaranteed the continuity of F? At this point, one may try to consider the second function $g(x) = x$, which, of course, is continuous on $[a,b]$. We ask : Was it the continuity of g that guaranteed the continuity of F? This is precisely the very substance of the following discussion.

Before we proceed let us mention a particular example of a discontinuous primitive.

Example 2.2 Let $A \in L(X,Y;Z)$. Let a_1, a_2, a_3 be distinct vectors in X, b_1, b_2, b_3 be distinct vectors in Y, and let

$$f(\xi) = \begin{cases} a_1, & \xi \in [0,1) \\ a_2, & \xi = 1 \\ a_3, & \xi \in (1,2] \end{cases}$$

and

$$g(\xi) = \begin{cases} b_1, & \xi \in [0,1) \\ b_2, & \xi = 1 \\ b_3, & \xi \in (1,2] \end{cases}$$

Then $\int_0^2 A(f,dg) = A(a_2, b_3 - b_1)$. If $F(x) = \int_a^x A(f,dg)$ for each $x \in [0,2]$, then

$$\|F(y) - F(1)\| = \|A(a_2, b_2 - b_1)\|$$

for all $0 \leq y < 1$, and

$$\|F(y) - F(1)\| = \|A(a_2, b_3 - b_2)\|$$

for all $1 < y \leq 2$. If $A(a_2, b_2 - b_1) \neq 0$, then any positive $\epsilon < \|A(a_2, b_2 - b_1)\|$ can be used to show that F is left-hand discontinuous at $\xi = 1$. And if $A(a_2, b_3 - b_2) \neq 0$, then any positive $\epsilon < \|A(a_2, b_3 - b_2)\|$ shows that F is right-hand discontinuous at $\xi = 1$.

The following theorem is a bilinear Stieltjes version of the (weaker) Henstock's lemma. A common procedure (see for reference [7] and [8]) and, in fact, the same lines of arguments prove the theorem.

Theorem 2.3 *Let $A \in L(X, Y; Z)$, and let $f : [a, b] \to X$ be Henstock-Stieltjes integrable with respect to A and $g : [a, b] \to Y$ on $[a, b]$ with primitive F. Given $\epsilon > 0$, there is a positive function $\delta(\xi)$ on $[a, b]$ such that for all δ-fine partial divisions $D = \{([u, v]; \xi)\}$ of $[a, b]$, we have*

$$\|(D) \sum \{F(u, v) - A(f(\xi), g(v) - g(u))\}\| < \epsilon.$$

Corollary 2.4 *Let $A \in L(X, Y; Z)$, and let $f : [a, b] \to X$ be Henstock-Stieltjes integrable with respect to A and $g : [a, b] \to Y$ on $[a, b]$ with primitive F. Then*

$$F(x) = \int_a^x A(f, dg) + F(a)$$

for all $x \in [a, b]$.

Proof: The integral $\int_a^x A(f, dg)$ exists for every $x \in [a, b]$ (see [1] and [7]). Let $x \in (a, b]$ and let $\epsilon > 0$. By Theorem 2, there exists a positive function $\delta(\xi)$ on $[a, b]$ such that for all δ-fine partial divisions $D = \{([u, v]; \xi)\}$ of $[a, b]$

$$\|(D) \sum \{F(u, v) - A(f(\xi), g(v) - g(u))\}\| < \epsilon.$$

Thus for all δ-fine divisions $D = \{([u,v];\xi)\}$ of $[a,x]$

$$\|(D)\sum A(f(\xi), g(v) - g(u)) - (F(x) - F(a))\}\| < \epsilon.$$

This means that

$$F(x) - F(a) = \int_a^x A(f, dg).$$ □

In view of Corollary 2.4, we remark that any two primitives of the same integral differ by a constant vector.

Example 2.5 Let $A \in L(X,Y;Z)$, let $f : [a,b] \to X$ be such that $f = 0$ almost everywhere in $[a,b]$ and let $g : [a,b] \to Y$ be absolutely continuous on $[a,b]$. Then $\int_a^b A(f,dg)$ exists and primitives are constant functions. To show this, let us proceed with a non-zero A. Let x be a fixed but arbitrary point in $[a,b]$. We claim that $\int_a^b A(f,dg)$ exists. Define

$$E = \{\xi \in [a,x] : f(\xi) \neq 0\},$$

and for each integer n, put

$$E_n = \{\xi \in E : n-1 < \|f(\xi)\| \leq n\}.$$

Then $E = \cup_n E_n$ and the measure of each E_n is $\mathbf{0}$.

Let $\epsilon > 0$. For every $n \in Z^+$, there is a $\delta_n > 0$ such that if $\{[a_i,b_i]\}$ is a finite or infinite sequence of non-overlapping intervals in $[a,x]$ with $\sum_i |b_i - a_i| < \delta_n$, then

$$\sum_i \|g(b_i) - g(a_i)\| < \frac{\epsilon}{n2^n\|A\|}.$$

For each $n \in Z^+$, there is a countable collection of disjoint open intervals $I_{n,1}, I_{n,2}, \cdots$ such that

$$E_n \subset \cup_i I_{n,i} \text{ and } \sum_i l(I_{n,i}) < \delta_n.$$

Define for each $\xi \in E_n$, a $\delta(\xi) > 0$ so that $(\xi - \delta(\xi), \xi + \delta(\xi)) \subset I_{n,i}$ for some i, and $\delta(\xi) = 1$ for every $\xi \in [a,b] \setminus E$. Let $D = \{([u,v];\xi)\}$ be any δ-fine division of $[a,x]$. For each $[u,v]$ in D, there is a pair (n,i) such that $[u,v] \subset I_{n,i}$. Therefore

$$\begin{aligned}
\|(D)\sum A(f(\xi), g(v)-g(u))\| &= \|(D)\sum_{\xi\in E} A(f(\xi), g(v)-g(u))\| \\
&\leq \|A\| \cdot \sum_n (\sum_{\xi \in E_n} \|f(\xi)\| \cdot \|g(v)-g(u)\|) \\
&\leq \|A\| \cdot \sum_n (n \sum_{\xi\in E_n} \|g(v)-g(u))\|) \\
&< \|A\| \cdot \sum_n (n \cdot \frac{\epsilon}{n2^n\|A\|}) \\
&= \epsilon.
\end{aligned}$$

This means that

$$\int_a^x A(f, dg) = 0.$$

Consequently,

$$\int_a^b A(f, dg) = 0,$$

and if F is a primitive, then , by Corollary 2.4,

$$F(x) = F(a) \;\; for \;\; all \;\; x \in [a,b]\,.$$ □

Theorem 2.6 *Let $A \in L(X,Y;Z)$, and let a bounded function $f : [a,b] \to X$ be Henstock-Stieltjes integrable with respect to A and $g : [a,b] \to Y$ on $[a,b]$ with primitive F. If g is absolutely continuous on $[a,b]$, then so is F.*

Proof: Assume $\|A\| \neq 0$, and let M be a positive bound for f. Given $\epsilon > 0$, let $\delta > 0$ be such that for all finite or infinite sequence of non overlapping intervals $\{[a_i, b_i]\}$ in $[a, b]$

$$\sum_i |b_i - a_i| < \delta \ \textit{implies} \ \sum_i \|g(b_i) - g(a_i)\| < \frac{\epsilon}{2M\|A\|}.$$

Now let $\{[a_i, b_i]\}$ be a finite or infinite sequence of non-overlapping intervals in $[a, b]$ such that $\sum_i |b_i - a_i| < \delta$. For every $i = 1, 2, \cdots$, there exists a positve function $\delta_i(\xi)$ on $[a, b]$ such that if

$$D_i : a_i = x_0^i < x_1^i < \cdots < x_{n_i}^i = b_i, \ \ \{\xi_1^i, \xi_2^i, \cdots, \xi_{n_i}^i\}$$

is any δ_i-fine division of $[a_i, b_i]$, then

$$\begin{aligned}
&|\|F(b_i) - F(a_i)\| - \|(D_i)\sum A(f(\xi), g(v) - g(u))\|| \\
&\leq \|F(b_i) - F(a_i) - (D_i)\sum A(f(\xi), g(v) - g(u))\| \\
&< \frac{\epsilon}{2^{i+1}}.
\end{aligned}$$

For every i fix a D_i, and consider the sequence $\{[x_{r-1}^i, x_r^i]\}_{r=1,2,\cdots,n_i}$. The intervals $[x_{r-1}^i, x_r^i]$, $r = 1, 2, \cdots, n_i$, $i = 1, 2, \cdots$ are non-overlapping. Since

$$\sum_{r=1,2,\cdots,n_i;i=1,2,\cdots} |x_r^i - x_{r-1}^i| = \sum_i \sum_{r=1}^{n_i} |x_r^i - x_{r-1}^i| = \sum_i |b_i - a_i| < \delta,$$

we have

$$\begin{aligned}
\sum_i \|F(b_i) - F(a_i)\| &\leq \sum_i \{(D_i)\sum \|A(f(\xi), g(v) - g(u))\| + \frac{\epsilon}{2^{i+1}}\} \\
&\leq M\|A\| \sum_i \{(D_i)\sum \|g(v) - g(u)\|\} + \sum_i \frac{\epsilon}{2^{i+1}} \\
&= M\|A\| \sum_{r=1,2,\cdots,n_i;i=1,2,\cdots} \|g(x_r^i) - g(x_{r-1}^i)\| + \sum_i \frac{\epsilon}{2^{i+1}} \\
&< M\|A\| \frac{\epsilon}{2M\|A\|} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

and the proof is complete. □

Theorem 2.7 *Let $A \in L(X, Y; Z)$, and let $f : [a, b] \to X$ be Henstock-Stieltjes integrable with respect to A and $g : [a, b] \to Y$ on $[a, b]$ with primitive F. If g is continuous on $[a, b]$, then F is continuous on $[a, b]$.*

Proof: Let $x \in [a, b]$. Assume $A \neq 0$ and take a positive number M where $\|f(x)\| < M$. If g is continuous on $[a, b]$, then given $\epsilon > 0$, there exists $\delta_1 > 0$ such that for any $y \in [a, b]$,

$$|x - y| < \delta_1 \;\; implies \;\; \|g(x) - g(y)\| < \frac{\epsilon}{2M\|A\|}.$$

By Theorem 1, there is a $0 < \delta_2 \leq \delta_1$ on $[a, b]$ such that for all δ_2-fine partial divisions $D = \{([u, v]; \xi)\}$ of $[a, b]$, we have

$$\|(D) \sum \{F(u, v) - A(f(\xi), g(v) - g(u))\}\| < \frac{\epsilon}{2}.$$

Now, let $y \in [a, b]$ be such that $x < y$ and $y - x < \delta_2(x)$. Then $\{([x, y]; x)\}$ is a δ_2-fine partial division of $[a, b]$, and so we have

$$\begin{aligned} \|F(x, y)\| &\leq \|F(x, y) - A(f(x), g(y) - g(x)\| + \|A(f(x), g(y) - g(x))\| \\ &< \frac{\epsilon}{2} + \|A\| \cdot \|f(x)\| \cdot \|g(y) - g(x)\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2M\|A\|} \|A\| \cdot \|f(x)\| \\ &< \epsilon. \end{aligned}$$

Similarly, if $y \in [a, b]$ is such that $y < x$ and $x - y < \delta_2(x)$, then $\{([y, x]; x)\}$ is a δ_2-fine partial division $D = \{([u, v]; \xi)\}$ of $[a, b]$, and

$$\|F(x) - F(y)\| < \epsilon.$$

Set $\delta = \delta_2(x)$ and we have that F is continuous at x. Since x is arbitrary, F is continuous on $[a, b]$. $\square$

Corollary 2.8 *Let $A \in L(X,Y;Z)$, and let $f:[a,b] \to X$ be Henstock-Stieltjes integrable with respect to A and to a regulated function $g:[a,b] \to Y$ on $[a,b]$. If F is a primitive of f with respect to A and g on $[a,b]$, then F is continuous on $[a,b]$ except possibly for a countable number of points.*

Proof: If g is regulated, then g has at most countably many points of discontinuity. The conclusion follows immediately from Theorem 5. □

Finally we turn to a mean convergence theorem. Versions of such in the classical case are thoroughly discussed in [**8**]. The one that follows is a Stieltjes version of the mean convergence theorem introduced by Cao in [**5**].

Theorem 2.9 *Let $A \in L(X,Y;Z)$, and let $g:[a,b] \to Y$ be of bounded variation. Let $\{f_n : [a,b] \to X\}$ be a sequence of Henstock-Stieltjes integrable functions with respect to A and g on $[a,b]$, each with primitive F_n, satisfying the following conditions:*

(i) $f_n(x) \longmapsto f(x)$ *for all* $x \in [a,b]$*;*

(ii) the primitives F_n converge pointwise to F on $[a,b]$;

(iii) $[a,b]$ is the union of sets $\{E_i\}$ such that for all i and for all $\epsilon > 0$, there exists an integer N_i, there exists $\delta(\xi) > 0$ on $[a,b]$ such that whenever $D = \{([u,v];\xi)\}$ is a δ-fine partial division of $[a,b]$ with $\xi \in E_i$, we have

$$\left\|(D)\sum\{F_n(u,v) - F_m(u,v)\}\right\| < \epsilon$$

for all $n, m \geq N_i$.

Then f is Henstock-Stieltjes integrable with respect to A and g on $[a,b]$ with primitive F.

Proof: Let $\epsilon > 0$. Condition (iii) implies that for all i, j, there exists $N_{i,j}$, there exists $\delta_i(\xi) > 0$ on $[a, b]$ such that whenever $D = \{([u, v]; \xi)\}$ is a δ_i-fine partial division of $[a, b]$ with $\xi \in E_i$, we have

$$\left\|(D)\sum\{F_n(u, v) - F_m(u, v)\}\right\| < \frac{\epsilon}{2^{i+j}}$$

for all $n, m \geq N_{i,j}$. In view of condition (ii), the above inequality implies that

$$\left\|(D)\sum\{F_n(u, v) - F(u, v)\}\right\| < \frac{\epsilon}{2^{i+j}}$$

for all $n \geq N_{i,j}$.

For each i, we may choose $\{F_{N_{i+1,j}} : j = 1, 2, \cdots\}$ as a subsequence of $\{F_{N_{i,j}} : j = 1, 2, \cdots\}$. Set $N(i) = N_{i,i}$, and consider the sequence $\{F_{N(i)} : i = 1, 2, \cdots\}$.

Let $E_1^* = E_1$ and $E_i^* = E_i - \bigcup_{k=1}^{i-1} E_k$ for $i > 1$. Then the sets E_i^* are pairwise disjoint whose union is $[a, b]$. Now condition (i) implies that for each $\xi \in E_i^*$, there exists a positive integer $m(\epsilon, \xi) \geq N(i)$ such that

$$\left\|f_{m(\epsilon,\xi)}(\xi) - f(\xi)\right\| < \epsilon.$$

We may choose $m(\epsilon, \xi) = N(j)$ for some $N(j) \geq N(i)$.

Since f_n is Henstock-Stieltjes integrable with respect to A and g on $[a, b]$, with primitive F_n, by Theorem 2, there exists a $\delta_n^*(\xi) > 0$ such that

$$\left\|(D)\sum\{A(f_n(\xi), g(v) - g(u)) - F_n(u, v)\}\right\| < \frac{\epsilon}{2^n}$$

for all δ_n^*-fine partial divisions $D = \{([u, v]; \xi)\}$ of $[a, b]$.

Put, for each $\xi \in E_i^*$,

$$\delta(\xi) = min\{\delta^*_{m(\epsilon,\xi)}(\xi), \delta_i(\xi)\},$$

and let $D = \{([u,v];\xi)\}$ be a δ-fine division of $[a,b]$. Then

$$\begin{aligned}
&\left\|(D)\sum\{A(f(\xi),g(v)-g(u))-F(u,v)\right\| \\
&\leq \left\|(D)\sum\{A(f(\xi)-f_{m(\epsilon,\xi)}(\xi),g(v)-g(u))\}\right\| + \\
&\quad + \left\|(D)\sum\{A(f_{m(\epsilon,\xi)}(\xi),g(v)-g(u))-F_{m(\epsilon,\xi)}(u,v)\}\right\| + \\
&\quad + \left\|(D)\sum\{F_{m(\epsilon,\xi)}(u,v)-F(u,v)\right\|.
\end{aligned}$$

Now, for the middle term in the right hand of the inequality, we decompose D into the union of sets $D_{m(\epsilon,\xi)}$, where $D_{m(\epsilon,\xi)}$ is the set of all interval-point pairs in D associated with the integer $m(\epsilon,\xi)$. Note that

$$\left\|(D_{m(\epsilon,\xi)})\sum\{A(f_{m(\epsilon,\xi)}(\xi),g(v)-g(u))-F_{m(\epsilon,\xi)}(u,v)\}\right\| < \frac{\epsilon}{2^{m(\epsilon,\xi)}},$$

and thus

$$\left\|(D)\sum\{A(f_{m(\epsilon,\xi)}(\xi),g(v)-g(u))-F_{m(\epsilon,\xi)}(u,v)\}\right\| < \sum_{n=1}^{\infty}\frac{\epsilon}{2^n}.$$

To get the last term, for each i, let $G_{m(\epsilon,\xi)}$ be the set of all interval-point pairs $([u,v];\xi)$ in D where $\xi \in E_i^*$ and ξ is associated with the integer $m(\epsilon,\xi)$. Note also that

$$\begin{aligned}
\left\|\sum_{\xi\in E_i^*}\{F_{m(\epsilon,\xi)}(u,v)-F(u,v)\}\right\| &\leq \sum_{m(\epsilon,\xi);\xi\in E_i^*}\left\|(G_{m(\epsilon,\xi)})\sum\{F_{m(\epsilon,\xi)}(u,v)-F(u,v)\}\right\| \\
&< \sum_{m(\epsilon,\xi);\xi\in E_i^*}\frac{\epsilon}{2^{i+j}},
\end{aligned}$$

where $N(j) = m(\epsilon,\xi)$ and $N(j) \geq N_{i,j}$. Thus

$$\begin{aligned}
\left\|(D)\sum\{A(f(\xi),g(v)-g(u))-F(u,v)\right\| &< 2\epsilon\cdot Var(g;[a,b]) + \sum_{n=1}^{\infty}\frac{\epsilon}{2^n} + \sum_{i,j}\frac{\epsilon}{2^{i+j}} \\
&= \epsilon\cdot\{2Var(g;[a,b])+2\}.
\end{aligned}$$

This means that $(HS)\int_a^b A(f,dg) = F(a,b) = \lim_n F_n(a,b)$. □

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