# An Introduction to Graph Folding

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**Abstract**

Graph folding is a unary operation introduced by Gervacio [6] in 1992 which he later modified in 1999.

This operation induces in a very natural way a partitioning of the vertex set of a graph into pairwise linked and independent sets. In this paper we give results involving the folding of regular graphs. Here, too, we show that the famous Petersen graph folds only into the complete graphs $K_3, K_4$, and $K_5$. In addition, among others, we will show that $max\{t|\ K_t \in F(G)\} \leq \left\lfloor \frac{1+\sqrt{1+4rn}}{2} \right\rfloor$, where $G$ is a connected $r$-regular graph of order $n$.

Furthermore, this paper enumerates some relationships of folding graphs to several graph invariants such as chromatic number, span, independence number, domination number, maximum degree, and dimension.

**Keywords:** graph, folding, chromatic number, span, dimension

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# 1 Introduction

We shall only consider connected simple graphs here, i.e., finite graphs without multiple edges and loops. *Folding a graph* simply means identifying two non-adjacent vertices of a graph and this can be repeated iteratively any number of times until there does not exist such pair of non-adjacent vertices.

A master's thesis [12] and a doctoral dissertation [9] were completed and successfully defended in 1995 and 2002, respectively. Furthermore, in August 2001 Gervacio presented a paper on this topic at a Seminar-Lecture series at University of Santo Thomas, Manila, sponsored by the Mathematical Society of the Philippines, National Capital Region Chapter.

In 1992, Gervacio [6] studied folding of graphs using a different construction. In that study, a folding of a connected graph $G$ is defined as the simple graph obtained from $G$ by identifying as one vertex two non-adjacent vertices and then reducing multiple edges formed to single edges. In contrast, this present study defines folding similar to the definition mentioned above with the added condition that two non-adjacent vertices to be identified must have a common neighbor.

Some of the results obtained from Gervacio's work [6] included the relationship of folding graphs with concepts of a perimeter and span of graphs. Formulas for the span and perimeter of a connected graph were also derived.

In her master's thesis, Vega [12] made further studies of folding graphs using the definition used in [6]. Results obtained include, among others, the determination of the *maximum-folding* and *minimum-folding* of a graph $G$, which are denoted by $mxf(G)$ and $mnf(G)$, respectively. Here $mxf(G)$ is defined as the $k$-folding of a graph $G$, where $k$ is the largest non-negative integer such that the resulting graph is a complete graph of minimum order while $mnf(G)$ is defined as the $k$-folding of $G$, where $k$ is the smallest non-negative integer for which the resulting complete graph is of maximum order.

As a matter of comparison, it should be stressed here that the folding used in this study may also be referred to as restricted folding while that in [6] and [12] is a non-restricted or free folding. Notice that in a restricted folding we identify two distinct vertices of a graph whose distance is 2 while in free folding we identify two distinct vertices whose distance is greater than or equal to 2.

Buckley and Harary [1] stated that Moon and Mosher proved that almost all graphs have diameter 2. Hence, under this construction, we may then be able to fold almost all graphs.

Observe further that a restricted folding is a free folding but not conversely. For instance, it will be shown that the cycle $C_6$ of order 6 restrictly folds uniquely into the complete graph $K_2$; while the same graph can be freely folded into $K_3$. In fact, it was shown in [12] that $mnf(C_6) = K_3$. However, $mxf(G) = K_2$ also. Thus, for an arbitrary graph $G$ it is not always true that we can obtain $mnf(G)$ using the folding used in this present study (restricted, as we referred it here).

It should be noted that in the cases of folding the wheel, fan, biwheel and bifan, the same minimum foldings were obtained using these two constructions of folding. This is expected because of the structures of these graphs.

Recently, Gervacio [7, 8] made separate investigations related to this study using restricted folding. Some of the results of these works have relevance on this study. One result obtained is a characterization theorem for folding graphs. Other important results are the partitioning the vertex-sets of the path and cycle into pairwise linked and independent sets. These results enable us to determine the exact orders of the maximum complete graphs into which the wheel and the fan can be folded.

# 2 The Idea of Graph Folding, Preliminary Concepts and Results

Unless stated otherwise, graphs will be denoted by capital letters such as $G, H$, etc. The set of vertices of $G$ and the set of edges of $G$ will be denoted by $V(G)$ and $E(G)$, respectively. Basic graph-theoretic concepts are assumed to be understood by the reader; otherwise, the readers may refer to the books by Chartrand [2], Foulds[3], and Harary [10]. New concepts will be defined formally.

**Definition 2.1** Let $G$ be a connected graph and let $x$ and $y$ be non-adjacent vertices with at least one common neighbor. We define the *xy-folding of G* (or simply the *folding of G* when no confusion can result), denoted by $f(G; x, y)$ (or, $f(G; y, x)$), to be the graph obtained from $G$ by identifying the vertices $x$ and $y$ and reducing multiple edges to single edges.

**Notation:** Let $x_1, x_2, \ldots, x_n$ be vertices of a graph $G$. The notation $\{x_1, x_2, \ldots$ means that $x_1, x_2, \ldots, x_n$ are identified.

**Example 2.2** The graph $G$ in Figure 1 has two vertices $x$ and $y$ with common neighbors $u$ and $v$. By identifying $x$ and $y$ and reducing the resulting double edges into simple edges, we get the graph $f(G, x, y)$. Note that $u$ and $z$ cannot be identified since they do not have a common neighbor.

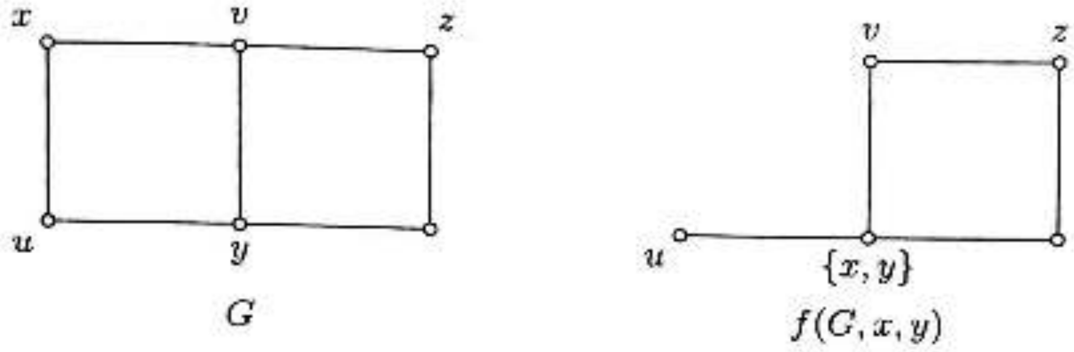


Figure 1: Folding of a Graph $G$ into $f(G, x, y)$.

Observe, further, that if a connected graph is not complete, then it always has two non-adjacent vertices that have a common neighbor. Thus, any connected non-complete graph can undergo a sequence of folding until we obtain a complete graph. Obviously, no loops can occur in any series of folding by definition.

**Notation:** We shall denote by $F(G)$ the set of all non-isomorphic complete graphs obtainable from a connected graph $G$ by a sequence of folding. That is,

$$F(G) = \{K_p |\ G \textit{ can be folded into } K_p\}.$$

If $F(G)$ has only one element, say $K_s$, then we shall simply write $F(G) = K_s$ instead of $F(G) = \{K_s\}$. For instance, it is obvious that $F(K_n) = K_n,\ \forall n$.

Let us fold further the graph $G$ in Figure 1 until we get a complete graph. (Please refer to Figure 2)

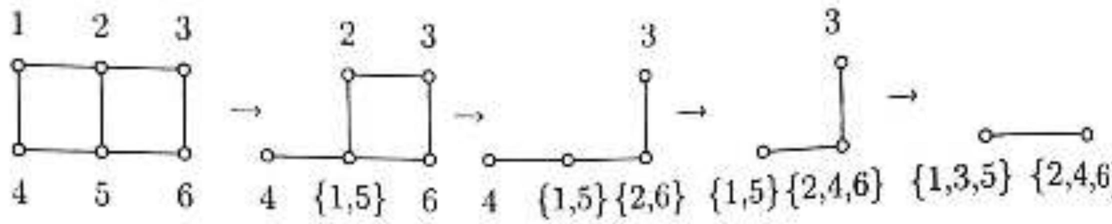


Figure 2: Folding a graph into a complete graph.

The graph was folded into the complete graph $K_2$. It was shown in [9] that a connected graph $G$ folds uniquely into $K_2$ if and only if $G$ is bipartite. In view of this result, we have $F(G) = K_2$ since the graph $G$ in Figure 1 is bipartite.

Let us observe that each of the two vertices of $K_2$ was obtained by identifying some vertices of the original graph. One of the vertices of $K_2$ was obtained by identifying the vertices 1, 3 and 5 while the other vertex was obtained by the vertices 2, 4, and 6. Observe that the sets $\{1,3,5\}$ and $\{2,4,6\}$ are both independent and pairwise linked sets which partitions $V(G)$. This observation is true for all graphs according to [9].

Indeed there are graphs $G$ for which $F(G)$ contains more than one complete graph as can be seen in the following example.

**Example 2.3** We show that $F(W_6) = \{K_3, K_4\}$.

*Solution*: By definition, $W_6 = C_6 + K_1$. In general, $W_n$ is commonly called the *wheel* of order $n+1$ while $C_n$ is called the *cycle* of order $n$. The graph of $W_6$ is shown in Figure 3.

We now fold this graph in two ways:

Case 1. If we choose $x$ and $y$ to be vertices whose distance in $C_6$ is 3, and then continue folding, we eventually get $K_4$.

Case 2. If in the first folding we choose two alternate vertices $x$ and $y$ in $C_6$ and then continue folding, we eventually get $K_3$.

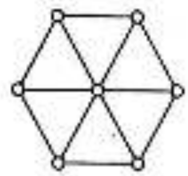

Figure 3: The Wheel $W_6$

**Example 2.4** We invite the interested reader to verify that $F(F_8) = \{K_3, K_4, K_5\}$, where $F_8$ is the fan of order 9.

We end this section by stating the following results which will prove relevant later on.

**Theorem 2.5** [9] *If $K_p \in F(G)$, then $V(G)$ can be partitioned into $p$ independent sets $S_1, S_2, \ldots, S_p$ which are pairwise linked.*

**Theorem 2.6** [2] *Let $G$ be a graph of order $n$ and $v \in V(G)$. If $deg_G(v) = r$, then $deg_{\overline{G}}(v) = n - r - 1$, where $\overline{G}$ is the complement of $G$. Therefore, $\overline{G}$ is regular if and only if $G$ is regular.*

# 3 Folding Regular Graphs

This section deals on the folding of some regular graphs.

A graph $G$ is called *regular* if all the vertices of $G$ are of equal degree. If $\forall x \in G, deg(x) = r$, we say that $G$ is an *$r$-regular graph*.

First, let us find all the distinct foldings of a very popular graph – the Petersen Graph which we shall denote by $G^*$. This graph is shown in Figure 3 together with a labelling of its vertices. Note that $G^*$ is 3-regular.

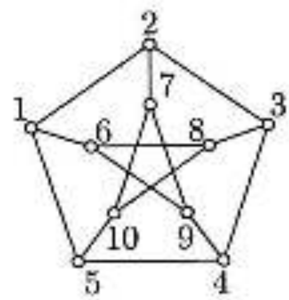


Figure 4: The Petersen graph $G^*$.

**Theorem 3.1** *The Petersen graph $G^*$ folds only into $K_3, K_4,$ and $K_5$, i.e., $F(G^*) = \{K_3, K_4, K_5\}$.*

*Proof*: Shown in Figure 3 is the Petersen graph with a labelling of its vertices. By a result in [9] mentioned earlier, $K_2 \notin F(G^*)$, since $G^*$ is not bipartite.

A 3-partition of $V(G^*)$ that folds $G^*$ into $K_3$ is the following:

$$S_1 = \{1, 8, 9\}, S_2 = \{3, 5, 7\}, S_3 = \{2, 4, 6, 10\}.$$

(Please refer to Figure 5.)

A 4-partition of $V(G^*)$ that folds $G*$ into $K_4$ is the following:

$$S_1 = \{1, 8, 4\}, S_2 = \{3, 5, 9\}, S_3 = \{2, 10\}, S_4 = \{6, 7\}.$$

A 5-partition of $V(G^*)$ that folds $G*$ into $K_5$ is the following:

$$S_1 = \{1, 4\}, S_2 = \{8, 9\}, S_3 = \{3, 10\}, S_4 = \{2, 6\}, S_5 = \{5, 7\}.$$

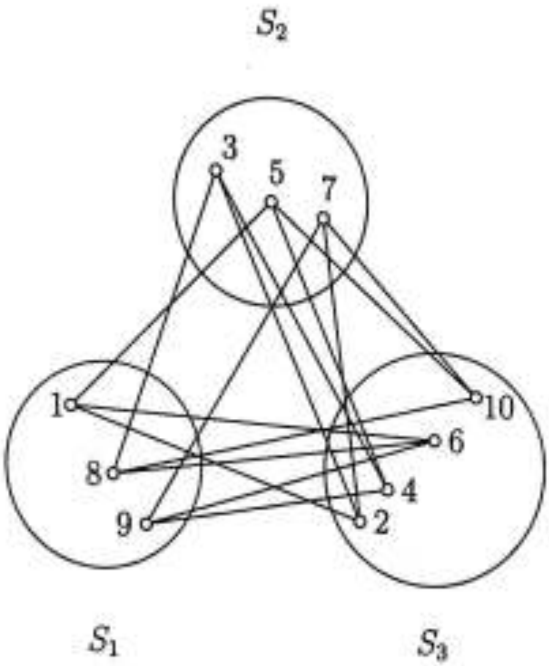


Figure 5: Partitioning $V(G^*)$ that folds $G^*$ into $K_3$.

Therefore we succeeded in folding the Petersen graph into $K_3, K_4$ and $K_5$.

We now claim that $G^*$ cannot be folded into $K_6$ and, hence into complete graphs of order $n > 6$. To see this, observe that each time we identify two non-adjacent vertices, the number of edges decreases. Note that $G^*$ has 15 edges and $K_6$ has $\binom{6}{2} = 15$ edges also. Hence, we cannot fold it into $K_6$.

We now conclude that $F(G^*) = \{K_3, K_4, K_5\}$. $\square$

**Theorem 3.2** *Let $G$ be a $k$-regular graph of diameter 2, where $k \geq 3$. If $G$ folds into $K_r$, $r \geq k+2$, then $G$ necessarily folds into $K_{r-1}$. In particular, any 3- regular graph of diameter 2 that folds into $K_r$, where $r \geq 5$, necessarily folds into $K_{r-1}$.*

*Proof*: Let $S_1, S_2, \ldots, S_r$ be the independent and pairwise linked sets induced by a folding of the $k$-regular graph $G$ of diameter 2 into $K_r$. Let $x \in V(G)$. Without loss of generality, let $x \in S_1$. Then $deg(x) = k \leq r - 2$. Thus $x$ is adjacent to at most $r-2$ vertices. But there are $r-1$ sets $S_2, S_3, \ldots, S_r$ and $r - 1 > r - 2 \geq k$. Hence there exists some set $S_j$, $2 \leq j \leq r$, such that $x$ is not adjacent to any vertex in $S_j$. We can transfer $x$ to $S_j$ maintaining the independence of $S_j$. Thus, every vertex in $S_1$ can be transferred to some set among $S_2, S_3, \ldots, S_r$. This leaves us with $r-1$ sets $S_2, S_3, \ldots, S_r$ that are pairwise linked and independent.

Therefore $G$ can now be folded into $K_{r-1}$. □

**Theorem 3.3** *Let $G$ be a connected $r$-regular graph of order $n$. Then*

$$max\{t|\ K_t \in F(G)\} \leq \left\lfloor \frac{1+\sqrt{1+4rn}}{2} \right\rfloor.$$

*Proof*: The theorem is clearly true when $G = K_n$. So assume $G$ is an $r$-regular graph, where $r \neq n-1$. Then, $\forall x \in V(G)$, $deg(x) = r$.

Let $p = max\{t|\ K_t \in F(G)\}$. By Theorem 2.5, $V(G)$ can be partitioned into $p$- independent and pairwise linked sets $S_1, S_2, \ldots, S_p$. Without loss of generality, assume

$$k = |S_1| \leq |S_2| \leq \cdots \leq |S_p|.$$

Since $deg(x) = r$, $\forall x \in S_1$ and $S_1$ is linked to all the other $p-1$ sets $S_2, S_3, \ldots, S_p$, then,

$$rk \geq p - 1.$$

Furthermore, since the $S_i's$ partition $V(G)$, then,

$$n \geq pk.$$

From the last two inequalities above, we obtain

$$n \geq p(\frac{p-1}{k})$$

by eliminating $k$. Thus

$$\begin{aligned} nr &\geq p^2 - p \\ p^2 - p - nr &\leq 0 \\ p &\leq \frac{1+\sqrt{4rn+1}}{2} \\ p &\leq \left\lfloor \frac{1+\sqrt{1+4rn}}{2} \right\rfloor. \end{aligned}$$

Therefore

$$max\{t|\ K_t \in F(G)\} \leq \left\lfloor \frac{1+\sqrt{1+4rn}}{2} \right\rfloor.$$ □

**Theorem 3.4** *Let $G$ be a connected $r$-regular graph of order $n$ and such that $\widetilde{G}$ is also connected. Then*

$$max\{t|K_t \in F(\widetilde{G})\} \leq \left\lfloor \frac{1+\sqrt{1+4n(n-r-1)}}{2} \right\rfloor.$$

*Proof*: This follows from Theorems 3.3 and 2.6. □

# 4 Other Concepts and Initial Results

Here we introduce some relevant concepts as well as results obtained in [9] and other related works.

Other concepts which are not explicitly defined here may be found in [10] and [2].

Let $G$ be a graph.

The largest of the vertex degrees of $G$ is called the *maximum degree* of $G$ and is denoted by $\Delta(G)$ or simply $\Delta$. A *coloring* of $G$ is an assignment of colors to its vertices (exactly one color for each vertex) so that no two adjacent vertices in $G$ have the same color. A coloring of $G$ which assigns c colors to its vertices is called a c – *coloring*.

The *chromatic number* of $G$ is the minimum number, denoted by $\chi(G)$, for which $G$ has a $\chi$- coloring. Equivalently, $\chi(G)$ is the minimum number $n$ for which $G$ is an $n$-partite graph. Graph $G$ is *c-colorable* if $\chi(G)$ is less than or equal to $c$. In other words, if there exists a $c$- coloring of a graph $G$, then $G$ is $c$-colorable. A *clique* in $G$ is a maximal complete subgraph of $G$. The maximum order of a clique is called the *clique number* of $G$, and is denoted by $\omega(G)$.

A set $S$ of vertices in $G$ is *independent* if no two vertices of $S$ are adjacent in $G$. The maximum cardinality of an independent set of vertices in $G$ is called the *independence number* of $G$ and is denoted by $\alpha(G)$. $S$ is a *dominating set* if every vertex not in $S$ is adjacent to a vertex in $S$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in $G$.

$G$ is called a *unit graph* in the Euclidean space $R^d$ if there is a one-to-one mapping $\phi\colon V(G) \to R^d$ such that $|\phi(x) - \phi(y)| = 1$ whenever $[x, y] \in E(G)$. The mapping $\phi$ is called a *unit representation* of $G$ in $R^d$. The *dimension* of $G$, denoted by $dim(G)$, is the smallest integer $d$ such that $G$ is a unit graph in $R^d$.

Let $G$ be a unit graph in $R^d$. The *span* of $G$ in $R^d$, denoted by $span_d(G)$, is the real number $s$ such that for every $\epsilon > 0$, the following conditions are satisfied:

1. There exists a unit representation of $G$ in $R^d$ which is contained in some open ball of diameter $s + \epsilon$;

2. No unit representation of $G$ in $R^d$ is contained in any open ball of diameter $s$.

Let $S$ and $T$ be two disjoint independent sets of vertices of a graph. We say that *$S$ and $T$ are linked* if there is a vertex in $S$ that is adjacent to a vertex in $T$.

**Remark 4.1** [9] *If $K_p \in F(G)$,i.e., if $G$ can be folded into $K_p$, then $G$ is p-partite.*

**Lemma 4.2** [2] *For every graph $G$, $\gamma(G) \leq \alpha(G)$.*

**Lemma 4.3** [3] *If $G$ is a graph of order $n$ and chromatic number $\chi(G)$. Then*

$$\chi(G)\alpha(G) \geq n.$$

**Lemma 4.4** [2] *Let $G$ be a connected graph with maximum degree $\Delta = \Delta(G)$. Then*

*1.* $\omega(G) \leq \chi(G) \leq 1 + \Delta$;

*2.* $\chi(G) \leq \Delta$ *if and only if $G$ is neither a complete graph nor an odd cycle.*

Part (2) is called Brook's Theorem.

**Lemma 4.5** [5] *For any graph G,*

$$dim(G) \leq 2\chi(G).$$

**Theorem 4.6** [11] *For any graph G with chromatic number* $\chi(G)$,

$$span_{2\chi(G)}(G) \leq \sqrt{\frac{2[\chi(G)-1]}{\chi(G)}}.$$

## 5 Relation of Folding with Some Graph Invariants

Let $K_p \in F(G)$. By Lemma 4.3, $\alpha(G)\chi(G) \geq n$. Hence

$$p \leq n \leq \alpha(G)\chi(G).$$

Therefore

$$p \leq \alpha(G)\chi(G).$$

This establishes the following:

**Remark 5.1** Let $G$ be a graph of order $n$ with independence number $\alpha(G)$ and chromatic number $\chi(G)$. If $K_p \in F(G)$, then

$$p \leq \alpha(G)\chi(G).$$

Note that equality holds when $G = K_n$; for then $\chi(G) = n$, $\alpha(G) = 1$, and $n = p$.

**Theorem 5.2** *Let $G$ be a graph with chromatic number $\chi(G)$. Then*

$$min\{r|\ K_r \in F(G)\} \geq \chi(G).$$

*Proof*: Let $p = min\{r|\ K_r \in F(G)\}$. Then $K_p \in F(G)$. Thus $G$ is a $p$-partite graph by Remark 4.1. Let $S_1, S_2, \ldots, S_p$ be a $p$-partition of $V(G)$. By Theorem 2.5, these sets are pairwise linked sets. Then $G$ is $p$-colorable since each set $S_i$ may receive one color different from the others. Note that in getting the chromatic number $\chi(G)$, we actually partitioned $V(G)$ into independent sets $S_1, S_2, \ldots, S_{\chi(G)}$ which are not necessarily mutually linked.

It follows that $p \geq \chi(G)$, by definition of $\chi(G)$.

Therefore

$$min\{r|\ K_r \in F(G)\} \geq \chi(G). \qquad \square$$

**Corollary 5.3** *Let $G$ be a graph with chromatic number $\chi(G)$. Then*

$$min\{r|\ K_r \in F(G)\} \geq \left\lceil \frac{2}{2 - [span_{2\chi(G)}(G)]^2} \right\rceil,$$

*where* $2 \neq [span_2\chi(G)]^2$.

*Proof*: Let $p = min\{r|\ K_r \in F(G)\}$. Using Theorem 4.6, we have the following

$$span_{2\chi(G)}(G) \leq \sqrt{\frac{2[\chi(G) - 1]}{\chi(G)}}$$

$$\chi(G)[span_{2\chi(G)}(G)]^2 \leq 2\chi(G) - 2$$

$$2 \leq \chi(G)\left\{2 - [span_{2\chi(G)}(G)]^2\right\}$$

$$\chi(G) \geq \frac{2}{2 - \left[span_{2\chi(G)}(G)\right]^2}.$$

By Theorem 5.2,

$$p = min\{r|\ K_r \in F(G)\} \geq \frac{2}{2 - \left[span_{2\chi(G)}(G)\right]^2}.$$

Hence,

$$p = min\{r|\ K_r \in F(G)\} \geq \left\lceil \frac{2}{2 - \left[span_{2\chi(G)}(G)\right]^2} \right\rceil.$$

This completes the proof. □

**Corollary 5.4** *Let $G$ be a graph with clique number $\omega(G)$. Then*

$$min\{r|\ K_r \in F(G)\} \geq \omega(G).$$

*Proof*: Let $p = min\{r|\ K_r \in F(G)\}$. By Theorem 5.2, $p \geq \chi(G)$. By Lemma 4.4, $\chi(G) \geq \omega(G)$. Thus, $p \geq \omega(G)$.

Therefore

$$min\{r|\ K_r \in F(G)\} \geq \omega(G). \qquad \square$$

Corollary 5.4 implies that $K_{\omega(G)}$ is a subgraph of $K_p$.

**Theorem 5.5** *Let $G$ be a graph of order $n$ with domination number $\gamma(G)$ and independence number $\alpha(G)$. Then*

$$min\{r|\ K_r \in F(G)\} > \left\lceil \frac{\gamma(G)}{\alpha(G)} \right\rceil.$$

*Proof*: By Lemma 4.2,

$$\gamma(G) \leq \alpha(G).$$

Thus

$$\frac{\gamma(G)}{\alpha(G)} \leq 1.$$

Clearly

$$\frac{\gamma(G)}{\alpha(G)} > 0.$$

Therefore

$$0 < \frac{\gamma(G)}{\alpha(G)} \leq 1.$$

But $min\{r|\ K_r \in F(G)\} \geq 2$; thus,

$$min\{r|\ K_r \in F(G)\} > \frac{\gamma(G)}{\alpha(G)}.$$

$$min\{r|\ K_r \in F(G)\} > \left\lceil \frac{\gamma(G)}{\alpha(G)} \right\rceil. \qquad \square$$

**Theorem 5.6** *Let $G$ be a graph of order $n$ with independence number $\alpha(G)$. Then*

$$min\{r|\ K_r \in F(G) \geq \left\lceil \frac{n}{\alpha(G)} \right\rceil.$$

*Proof*: Let $p = min\{r|\ K_r \in F(G)\}$. By Remark 4.1, $G$ is $p$-partite. Thus, $V(G)$ can be partitioned into $p$ independent sets $S_1, S_2, \ldots, S_p$. Thus we have the following disjoint union,

$$V(G) = S_1 \cup S_2 \cup \ldots \cup S_p.$$

Therefore

$$|V(G)| = n = |S_1| + |S_2| + \ldots + |S_p|.$$

Since $|S_i| \leq \alpha(G), \forall i$, then

$$n \leq \alpha(G) + \alpha(G) + \ldots + \alpha(G),$$

a sum of $p$ terms.

Hence $n \leq p\ \alpha(G)$, i.e., $p \geq \frac{n}{\alpha(G)}$. Since $p$ is a positive integer, we obtain

$$p \geq \left\lceil \frac{n}{\alpha(G)} \right\rceil.$$

Therefore

$$min\{r|\ K_r \in F(G)\} \geq \left\lceil \frac{n}{\alpha(G)} \right\rceil. \qquad \square$$

**Theorem 5.7** *Let $G$ be a graph of order $n$ with maximum degree $\Delta$. Then*

$$max\{r|\ K_r \in F(G)\} \leq \left\lfloor \frac{1 + \sqrt{1 + 4n\Delta}}{2} \right\rfloor.$$

*Proof*: Let $G$ be a graph of order $n$ and with maximum degree $\Delta$. Let $\cdot = max\{r|\ K_r \in F(G)\}$. By Theorem 2.5, $V(G)$ can be partitioned into

$q$ independent sets $S_1, S_2, \ldots, S_q$ which are pairwise linked. Without loss of generality, assume that

$$k = |S_1| \leq |S_2| \leq \cdots \leq |S_q|.$$

We claim that $k\Delta \geq q - 1$. Observe that the number of edges with one end vertex in $S_1$ is equal to

$$\sum_{x \in S_1} deg(x).$$

Since $S_1$ is linked to all the other sets $S_2, S_3, \ldots, S_q$, then

$$\sum_{x \in S_1} deg(x) \geq q - 1.$$

Because $\forall x \in S_1$, $deg(x) \leq \Delta$, then

$$\sum_{x \in S_1} deg(x) \leq k\Delta.$$

It follows that

$$k\Delta \geq q - 1.$$

This proves our claim.

Then we have the following inequalities

$$n \geq qk \qquad (1)$$

$$k\Delta \geq q - 1. \qquad (2)$$

From these two inequalities, we get

$$n \geq q(\frac{q-1}{\Delta})$$

by eliminating $k$. Thus

$$\begin{aligned} n\Delta &\geq q^2 - q \\ q^2 - q - n\Delta &\leq 0 \\ q &\leq \frac{1+\sqrt{1-4(1)(-n\Delta)}}{2} \\ q &\leq \frac{1+\sqrt{1+4n\Delta}}{2} \\ q &\leq \left\lfloor \frac{1+\sqrt{4n\Delta+1}}{2} \right\rfloor. \end{aligned}$$

Therefore

$$max\{r |\ K_r \in F(G)\} \leq \left\lfloor \frac{1+\sqrt{1+4n\Delta}}{2} \right\rfloor. \qquad \square$$

**Theorem 5.8** *Let $G$ be a graph. Then*

$$min\{r |\ K_r \in F(G)\} \geq \left\lceil \frac{dim(G)}{2} \right\rceil.$$

*Proof*: By Theorem 5.2 and Lemma 4.5, we have

$$min\{r |\ K_r \in F(G)\} \geq \chi(G)$$

and

$$\chi(G) \geq \frac{dim(G)}{2}.$$

Hence

$$min\{r |\ K_r \in F(G)\} \geq \frac{dim(G)}{2}.$$

Therefore

$$min\{r |\ K_r \in F(G)\} \geq \left\lceil \frac{dim(G)}{2} \right\rceil. \qquad \square$$

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