On Topological Properties Involving Semi-Open Sets

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Abstract

In this paper, the concept of s-connectedness will be introduced. Some results concerning the relationship between connectedness and s-connectedness will be given. In particular, it is shown that connectedness is weaker than s-connectedness. Characterizations of s-connectedness are also obtained.

Keywords: topology, semi-open, semi-continuity, s-connectedness

1 Introduction

The concepts of semi-open sets and semi-continuity in a topological space were introduced by Norman Levine in 1963 [5]. Apparently, the family of all semi-open sets in a topological space X contains all the open sets. Even though an arbitrary union of semi-open sets is semi-open, the class does not always form a topology on the underlying set. In particular, the intersection of two semi-open sets need not be semi-open.

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2 Definitions and Known Results

Definition 2.1. A space Y is ***connected*** if it is not the union of two non-empty disjoint open sets. Otherwise, Y is disconnected. A subset B of Y is connected if it is connected as a subspace of Y.

The proofs of the following results can be found in [2].

Theorem 2.2. *The only connected subsets of $\mathbb{R}$ having more than one point are $\mathbb{R}$ and the intervals (open, closed, or half-open).*

Theorem 2.3. *The following three properties are equivalent:*

(1) *Y is connected.*

(2) *The only subsets of Y both open and closed are $\varnothing$ and Y.*

(3) *No continuous $f : Y \to \mathbf{2}$ is surjective, where $\mathbf{2}$ is the space $X = \{0, 1\}$ with the discrete topology.*

Definition 2.4. Let X be a topological space. A subset O of X is ***semi-open*** if $O \subseteq cl(int(O))$. Equivalently, O is semi-open if there exists an open set G in X such that

$$G \subseteq O \subseteq cl(G) .$$

The family of semi-open sets in X is denoted by $\boldsymbol{SO(X)}$.

Definition 2.5. Let X and Y be topological spaces. A function $f : X \to Y$ is ***semi-continuous*** on X if $f^{-1}(O)$ is semi-open in X for every open set O in Y.

Definition 2.6. Let X and Y be topological spaces. A function $f : X \to Y$ is ***semi-continuous at p*** in X if for every semi-open subset V of Y with $f(p) \in V$, there exists a semi-open set U in X such that $p \in U$ and $f(U) \subseteq V$.

In [1], the authors proved the following:

Theorem 2.7. *f is semi-continuous if and only if it is semi-continuous at every element p of X.*

3 Results

This section presents the results on semi-continuity and s-connectedness. See below some characterizations involving these concepts.

Lemma 3.1. *Let X be an infinite set with the cofinite topology $\mathscr{T}_X$. Then $SO(X) = \mathscr{T}_X$.*

Proof. Since open sets are semi-open sets, $\mathscr{T}_X \subseteq SO(X)$. So it remains to show that $SO(X) \subseteq \mathscr{T}_X$. To this end, let $G \in SO(X) \backslash \mathscr{T}_X$. Then there exists $A \in \mathscr{T}_X \backslash \{\varnothing\}$ such that $A \subseteq G \subseteq cl(A)$; hence, $G^c \subseteq A^c$. Since A^c is finite, it follows that G^c is finite. Therefore $G \in \mathscr{T}_X$. Accordingly, $SO(X) = \mathscr{T}_X$. □

Theorem 3.2. *Let X be an infinite set with the cofinite topology and Y an arbitrary space. Then the following statements are equivalent:*

(a) *$f : X \to Y$ is semi-continuous;*

(b) *If O is open subset of Y, then $f^{-1}(O^c) = X$ or $f^{-1}(O^c)$ is finite.*

(c) *$f : X \to Y$ is continuous.*

Proof. (a):(b) Let O be an open subset of Y. Then by semi-continuity of f, $f^{-1}(O) \in SO(X)$; hence, $f^{-1}(O) \in \mathscr{T}_X$ by Lemma 3.1. This implies that

$$(f^{-1}(O))^c = f^{-1}(O^c)$$

is X or $f^{-1}(O^c)$ is finite. Hence, (b) holds.

(b):(c) Let O be an open set in Y. Then $f^{-1}(O) = \varnothing$ or $(f^{-1}(O))^c$ is finite, by assumption. Therefore $f^{-1}(O)$ is open in X; hence, f is continuous.

(c):(a) This is immediate from Lemma 3.1. □

Lemma 3.3. *Let $(X, \mathscr{T})$ be a topological space and Y an open subset of X. Then*

$$SO(Y) = \{G \in SO(X) : G \subseteq Y\} = T .$$

Proof. Suppose $O \in SO(X)$ and $O \subseteq Y$. Then there exists an open set G in X such that $G \subseteq O \subseteq cl(G)$. Since $O \subseteq Y$, it follows that $G \subseteq Y$; hence, $G = G \cap Y \in \mathscr{T}_Y$. Therefore,

$$G = G \cap Y \subseteq O = O \cap Y \subseteq Y \cap cl(G) ,$$

i.e.,

$$G \subseteq O \subseteq Y \cap cl(G) = cl_Y(G) .$$

This implies that $O \in SO(Y)$; hence

$$T = \{G \in SO(X) : G \subseteq Y\} \subseteq SO(Y) .$$

Next, let $O^* \in SO(Y)$. Then $O^* \subseteq Y$ and there exists $G^* \in \mathscr{T}_Y$ such that

$$G^* \subseteq O^* \subseteq cl_Y(G^*) .$$

It follows that there exists $G \in \mathscr{T}$ such that

$$G \cap Y \subseteq O^* \subseteq Y \cap cl(G^*) \subseteq cl(G^*) = cl(G \cap Y) .$$

Let $H = G \cap Y$. Since $Y \in \mathscr{T}$, $H \in \mathscr{T}$ and

$$H \subseteq O^* \subseteq cl(H) .$$

This shows that $O^* \in SO(X)$. Thus $SO(X) \subseteq T$. □

Theorem 3.4. *Let $f : X \to Y$ be a function and $X = X_1 \cup X_2$, where X_1 and X_2 are open subsets of X. If $f|_{X_1}$ and $f|_{X_2}$ are semi-continuous, then f is semi-continuous.*

Proof. Let $p \in X$ and let V be a semi-open subset of Y with $f(p) \in V$. Then $p \in X_1$ or $p \in X_2$. Without loss of generality, assume that $p \in X_1$. By semi-continuity of $f|_{X_1}$, there exists $U \in SO(X_1)$ with $p \in U$ such that $f(U) \subseteq V$. Consequently, by Lemma 3.3, there exists $U \in SO(X)$ with $p \in U$ such that $f(U) \subseteq V$. Therefore f is semi-continuous at p. Since p was arbitrarily chosen, it follows from Theorem 2.7 that f is semi-continuous on X. □

Theorem 3.5. *Let $X = X_1 \cup X_2$ and $f : X \to Y$. If $f|_{X_1}$ and $f|_{X_2}$ are semi-continuous at $p \in X_1 \cap X_2$, then f is semi-continuous at p.*

Proof. Let U be a semi-open set in Y containing $f(p)$. Since $f|_{X_1}$ and $f|_{X_2}$ are semi-continuous at $p \in X_1 \cap X_2$, there exist semi-open sets O_1 and O_2 in X_1 and X_2, respectively, such that $p \in O_1 \cap O_2$ and

$$f(O_1) \subseteq U , \ f(O_2) \subseteq U .$$

It follows that there exists semi-open set $O = O_1 \cup O_2$ in X such that $p \in O$ and

$$f(O) = f(O_1 \cup O_2) = f(O_1) \cup f(O_2) \subseteq U .$$

This shows that f is semi-continuous at p. □

Theorem 3.6. *If X is s-connected topological space, then X is connected.*

Proof. Suppose that X is an s-connected topological space. Then X is not the union of two non-empty disjoint semi-open sets. Since $\mathcal{T} \subseteq SO(X)$, X is not the union of two non-empty disjoint open sets. Thus X is connected. □

Remark 3.7. *The converse of Theorem* 3.6 *is not true.*

Consider the real line $\mathbb{R}$ with the usual topology. $\mathbb{R}$ is a connected space since $\mathbb{R}$ and $\varnothing$ are the only subsets of $\mathbb{R}$ which are both open and closed. Now, the interval $[a, +\infty)$ is semi-open in $\mathbb{R}$ since there exists an open set $(a, +\infty)$ in $\mathbb{R}$ such that

$$(a, +\infty) \subseteq [a, +\infty) \subseteq cl((a, +\infty)) = [a, +\infty) .$$

If follows that

$$\mathbb{R} = (-\infty, a) \cup [a, +\infty)$$

is the union of two non-empty disjoint semi-open sets, i.e., $\mathbb{R}$ is not s-connected. Thus, $\mathbb{R}$ is a connected topological space that is not s-connected.

Corollary 3.8. *The only s-connected subsets of $\mathbb{R}$ having more than one point are $\mathbb{R}$ and the intervals (open, closed, or half-open).*

Proof. This follows from Theorem 3.6 and Theorem 2.2.

Theorem 3.9. *The cofinite topology on X is s-connected.*

Proof. Suppose X is not s-connected, say $X = A \cup B$, where A and B are disjoint non-empty semi-open sets. Then A and B are open sets by Lemma 3.1; hence, X is disconnected. This contradicts the fact that cofinite topology on X is connected. □

Lemma 3.10. *Let X be any space and χ_A the characteristic function of $A \subseteq X$. Then χ_A is semi-continuous if and only if A is both semi-open and semi-closed.*

Proof. Suppose A is both semi-open and semi-closed. Let O be an open set in $\{0,1\}$. Then

$$\chi_A^{-1}(O) = \begin{cases} \varnothing & \text{if } O = \varnothing \\ X & \text{if } O = \{0,1\} \\ A & \text{if } O = \{1\} \\ A^c & \text{if } O = \{0\}. \end{cases}$$

Hence, $\chi_A^{-1}(O)$ is semi-open. Therefore, χ_A is semi-continuous.

Next, suppose that χ_A is semi-continuous. Let $O_1 = \{1\}$ and $O_2 = \{0\}$. Then O_1 and O_2 are open in $\{0,1\}$. Hence

$$\chi_A^{-1}(O_1) = A \text{ and } \chi_A^{-1}(O_2) = A^c$$

are semi-open in X. Thus, A is both semi-open and semi-closed. □

Theorem 3.11. *The following three properties are equivalent.*

(i) *Y is s-connected.*

(ii) *The only subsets of Y both semi-open and semi-closed are $\varnothing$ and Y.*

(*iii*) *No semi-continuous* $f : Y \to \mathbf{2}$ *is surjective, where* $\mathbf{2}$ *is the space* $X = \{0,1\}$ *with the discrete topology.*

Proof. (i):(*ii*) If G is both semi-open and semi-closed, where G is a non-empty proper subset of Y, then $G \cup G^c$ is a decomposition of Y; hence Y is not s-connected.

(*ii*):(*iii*) If $f : Y \to \mathbf{2}$ were a semi-continuous surjection, then

$$f^{-1}(\{0\}) \neq \varnothing, Y$$

(not Y for otherwise, $\exists\ y \in Y$ with $y \mapsto 1$ and $y \mapsto 0$ which is a contradiction). Since $\{0\}$ is open and closed in $\mathbf{2}$, $f^{-1}(\{0\})$ is both semi-open and semi-closed in Y, a contradiction to (*ii*).

(*iii*):(*i*) If $Y = A \cup B$, where A and B are disjoint non-empty semi-open sets, then A and B are also semi-closed. Consider the characteristic function of $A \subseteq Y$

$$\chi_A : A \to (\{0,1\}, \mathscr{D}) .$$

By Lemma 3.10, χ_A is semi-continuous. This is a contradiction to (*iii*). □

Theorem 3.12. *The semi-continuous image of an s-connected set is connected. That is, if* X *is s-connected and* $f : X \to Y$ *is semi-continuous, then* $f(X)$ *is connected.*

Proof. Suppose X is s-connected and $f : X \to Y$ is semi-continuous. If $f(X)$ were not connected, there would be, by Theorem 2.3, a continuous surjection $g : f(X) \to \mathbf{2}$. It follows that $g \circ f : X \to \mathbf{2}$ is semi-continuous on X. Let $z \in \{0,1\}$. Then $\exists\ y \in f(X)$ such that $g(y) = z$. Also, $\exists\ x \in X$ such that $f(x) = y$. Thus

$$g(f(x)) = (g \circ f)(x) = z .$$

This shows that $g \circ f$ is surjective, a contradiction to the fact that X is s-connected. □

Theorem 3.13. *Each semi-continuous real-valued function on an s-connected space X takes all values between any two that it assumes.*

Proof. Let $f : X \to \mathbb{R}$ be a semi-continuous function. Then $f(X)$ is a connected subset of $\mathbb{R}$ by Theorem 3.12. By Theorem 2.2, $f(X)$ is an interval. Thus if $f(x) = a$ and $f(x') = b$, where $a < b$, then $[a, b] \subseteq f(X)$. This implies that if $c \in [a, b]$, then $\exists\, z \in X$ such that $f(z) = c$. □

Theorem 3.14. *A discrete space having more than one point is never s-connected and that a space having the indiscrete topology is always s-connected.*

Proof. Suppose $|X| > 1$ and let $x \in X$. If $\mathscr{T} = \mathscr{D}$, then $\{x\}$ and $X \backslash \{x\}$ are disjoint non-empty open sets in $\mathscr{D}$ such that

$$X = \{x\} \cup (X \backslash \{x\})$$

Since every open set is semi-open, X is a union of disjoint non-empty semi-open sets. Thus X is not s-connected.

If $(X, \mathscr{T})$, $X \neq \varnothing$ and $\mathscr{T} = \mathscr{I}$, then $\varnothing$ and X are the only subsets of X that are both open and closed. This implies that $\varnothing$ and X are the only subsets of X that are both semi-open and semi-closed. Thus, by Theorem 3.11, X is s-connected. □

Theorem 3.15. *Let $\mathscr{T}$ and $\mathscr{T}_1$ be topologies on X. If $(X, \mathscr{T})$ is s-connected and $\mathscr{T}_1 \subseteq \mathscr{T}$, then $(X, \mathscr{T}_1)$ is s-connected.*

Proof. Let $(X, \mathscr{T})$ be s-connected. By Theorem 3.11, the only subsets of X that are both semi-open and semi-closed are $\varnothing$ and X. Since $\mathscr{T}_1 \subseteq \mathscr{T}$, the only subsets of X with respect to $\mathscr{T}_1$ that are both semi-open and semi-closed are $\varnothing$ and X. By Theorem 3.11, $(X, \mathscr{T}_1)$ is s-connected. □

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