On the Geodetic Cover and Geodetic Basis of the Corona of Graphs

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Abstract

Any path of length $\text{dist}_G(u, v)$ is a u-v *geodesic* in G, where u and v are vertices of a connected graph G. The set $I[u, v]$ denotes the closed interval consisting of u, v and all vertices lying on some u-v geodesic. If $A \subseteq V(G)$, then $I[A]$ is the union of all sets $I[u, v]$ for all $u, v \in A$. If $I[A] = V(G)$, then A is a *geodetic cover*. A geodetic cover of minimum cardinality is called a geodetic basis.

The *corona* of graphs G and H is the graph obtained by taking a copy of G of order n and n copies of H, and then joining the *ith* vertex of G to every vertex in the *ith* copy of H. In this paper, we give the order of the geodesic basis of the corona of two connected graphs.

Keywords: graph, corona, geodesic, geodetic cover, geodetic basis

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1 Preliminaries

Let G be a connected simple graph and $u, v \in V(G)$. The distance $d_G(u,v)$ between u and v in G is the length of a shortest u-v path $P(u,v)$ in G. Such u-v path of length $d_G(u,v)$ is called a u-v *geodesic*. The couple $(V(G), d_G)$, where $V(G)$ is the vertex set of G, is a metric space. The symbol $I_G[u,v]$ is used to denote the set consisting of u, v and all vertices lying on some u-v geodesic in G. A subset C of $V(G)$ is *convex* if for every two vertices $u, v \in C$, $I_G[u,v] \subseteq C$. A *geodetic cover* of G is a subset A of $V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in A; that is, $I_G[A] = V(G)$, where $I_G[A] = \bigcup_{u,v \in A} I[u,v]$. A *geodetic number* $g(G)$ of G is the minimum order of its geodetic covers, and any cover of order $g(G)$ is called a *geodetic basis*. It can easily be verified that $A \subseteq I_G[A]$ and that $I_G[A] = A$ if and only if A is convex. When no confusion arises, we simply refer $I_G[A]$ as $I[A]$.

The concept of geodetic basis and geodetic number of a graph was investigated in [1], [2], [3], [4], [5], [6], [7], [8] and [11]. For other graph theoretic terms, which are assumed here, readers are advised to see [10].

The following result can be found in [2] but no proof is given.

Lemma 1.1 *Let G be a connected graph. Then $g(G) = |V(G)|$ if and only if G is a complete graph.*

Proof: Suppose G is a complete graph. If $S \subseteq V(G)$, then S induces a complete subgraph of G. It follows that S is convex in G and hence $I[S] = S$. This implies that $I[S] = V(G)$ if and only if $S = V(G)$. Therefore $g(G) = |V(G)|$.

Conversely, suppose that $g(G) = |V(G)|$. Assume further that G is not complete. Then there exist $a, b \in V(G)$ such that $d(a,b) > 1$. Let c be a vertex in some a-b geodesic that is distinct from a and b. Set $S = V(G)\backslash\{c\}$. Clearly, S is not convex in G; that is, $I[S] \neq S$. Also, because $c \in I[a,b]$, it follows that $c \in I[S]$. Thus, $I[S] = V(G)$; that is, S is a geodetic cover in G. Therefore, $g(G) \leq |S| = |V(G)| - 1$. This contradicts our assumption. Therefore, G is a complete graph. □

2 Corona of Graphs

Definition 2.1 [2] The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking a copy of G of order n and n copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H.

Remark 2.2 If $H \cong K_1$, then the above definition is the same as that of [9].

We denote by $u + H_u$ a subgraph of $G \circ H$ obtained by joining the vertex u of G and all the vertices of a copy H_u of H. Thus, $u + H_u = \langle\{u\}\rangle + H_u$ is an induced subgraph of $G \circ H$.

Remark 2.3 Let G be a connected graph. Then the following holds in $G \circ H$:

1. $\cup_{u \in V(G)} V(H_u) \cup V(G) = V(G \circ H)$;
2. $V(u + H_u) \cap V(v + H_v) = \emptyset$ and
   $E(u + H_u) \cap E(v + H_v) = \emptyset$ for all distinct vertices u, v in G;

3. If G is trivial, then $G \circ H \cong H + K_1$; and

4. If G is non-trivial, then every vertex of G is a cut vertex in $G \circ H$.

Theorem 2.4 [10] *Let G be a (non-trivial) connected graph. Then v is a cut vertex in G if and only if there exists a partition of $V(G)\backslash\{v\}$ into subsets U and W such that for all $u \in U$ and $w \in W$, v is on every u-w path.*

Theorem 2.5 *Let G be a non-trivial connected graph. If $U, W \subseteq V(G)\backslash\{v\}$ are partitions of $V(G)\backslash\{v\}$ such that v is on every u-w path for all $u \in U$ and $w \in W$, then $W \cup \{v\}$ and $U \cup \{v\}$ are convex sets in $V(G)$.*

Proof: Since v is on every u-w path for all $u \in U$ and $w \in W$, v is in a u-w geodesic for all $u \in U$ and $w \in W$.

Suppose $C = W \cup \{v\}$ is not convex. Then there exist $x, y \in C$ and $z \notin C$ such that z is in some x-y geodesic, say

$$P(x, y) = [x_1, x_2, \ldots, x_r, z, y_1, y_2, \ldots, y_s],$$

where $x_1 = x, y_s = y$ and $r \geq 1$, $s \geq 1$.

Case 1. Suppose one of the vertices x and y is v, say, $x = v$. Since $z \in U$, $y \in W$ and $x \neq z$, the distinct subpaths $[x_1, x_2, \ldots, x_r, z]$ and $[z, y_1, y_2, \ldots, y_s]$ of $P(x, y)$ contain the vertex v. This is not possible because $P(x, y)$ is a path.

Case 2. Suppose none of the vertices x and y is v. Then $x, y \in W$. This implies that both paths $[x_1, x_2, \ldots, x_r, z]$ and $[z, y_1, y_2, \ldots, y_s]$ contain the vertex v. Again, this is impossible.

Therefore, $W \cup \{v\}$ is a convex set in $V(G)$. Similarly, $U \cup \{v\}$ is a convex set in $V(G)$. □

Since $V(H_u)$ and $V(G \circ H) \backslash V(u + H_u)$ are partitions of $V(G \circ H) \backslash \{u\}$ and $\{u\}$ is a cut vertex of $V(G \circ H)$ for all $u \in V(G)$, we have the following consequence of Theorem 2.5.

Corollary 2.6 *Let G be a non-trivial connected graph. Then $V(u + H_u)$ and $V(G \circ H) \backslash V(H_u)$ are convex sets in $V(G \circ H)$ for every $u \in V(G)$.*

Lemma 2.7 *Let G be a non-trivial connected graph. If A and B are non-empty subsets of $V(G)$ with $A \subseteq B$, then $I[A] \subseteq I[B]$.*

Proof: Let $x \in I[A]$. Then there exist $u, v \in A$ such that $x \in I[u, v]$. Since $A \subseteq B$, there exist $u, v \in B$ such that $x \in I[u, v]$. Thus, $x \in I[B]$. Therefore, $I[A] \subseteq I[B]$. □

Theorem 2.8 *Let G be a non-trivial connected graph. If $U, W \subseteq V(G) \backslash \{v\}$ are partitions of $V(G) \backslash \{v\}$ such that v is on every u-w path for all $u \in U$ and $w \in W$ and if A is a geodetic cover in G, then $A \cap U \neq \emptyset$ and $A \cap W \neq \emptyset$.*

Proof: Suppose A is a geodetic cover in G and suppose $A \cap U = \emptyset$. Then $A \subseteq V(G) \backslash U$. Thus, $I[A] \subseteq I[V(G) \backslash U]$ by Lemma 2.7. Since A is a geodetic cover in G and $V(G) \backslash U = W \cup \{v\}$ is convex by Theorem 2.5, we have

$$I[A] = V(G) \subseteq I[V(G) \backslash U] = V(G) \backslash U .$$

This is not possible because $U \neq \emptyset$. Therefore, $A \cap U \neq \emptyset$. Similarly, $A \cap W \neq \emptyset$. □

Corollary 2.9 *Let G be a nontrivial connected graph and S a geodetic cover of $G \circ H$. Then $S \cap V(H_u) \neq \emptyset$ for all $u \in V(G)$.*

Theorem 2.10 *Let G be a non-trivial connected graph and let A be a geodetic basis of G. If v is a cut vertex in G, then $v \notin A$.*

Proof: Suppose A is a geodetic basis of G. Further, suppose v is a cut vertex in G and $v \in A$. By Theorem 2.4, there exists a partition of $V(G)\backslash\{v\}$ into subsets U and W such that v is on every u-w path for all $u \in U$ and $w \in W$. This implies that v lies on a u-w geodesic in G for all $u \in U$ and $w \in W$. Let $B = A\backslash\{v\}$. Consider the following cases:

Case 1. Suppose $x = v$. Since A is a geodetic cover of G, $A \cap U \neq \emptyset$ and $A \cap W \neq \emptyset$ by Theorem 2.8. Pick $y \in A \cap U$ and $z \in A \cap W$. Then $x \in I[y, z]\backslash\{y, z\}$, where $y, z \in B$. Hence $x \in I[B]$.

Case 2. Suppose $x \in A\backslash\{v\} = B$. Since $B \subseteq I[B]$, it follows that $x \in I[B]$.

Case 3. Suppose $x \in V(G)\backslash A$. Then there exist $y, z \in A$ such that $x \in I[y, z]$, say $P(y, z) = [y, y_1, y_2, \ldots, y_r, z]$ is a y-z geodesic, where $x = y_k$ $(1 \leq k \leq r)$. If $y, z \in B$, then $x \in I[B]$. So, suppose without loss of generality, that $y = v$. Then either $x, z \in U$ or $x, z \in W$. Assume that $x, z \in U$. Pick $y^* \in A \cap W$ and let $P(y^*, x) = [y^*, x_1, x_2, \ldots, x_t, x]$ be a y^*-x geodesic. Then $y = v$ is a vertex in $P(y^*, x)$. It follows that

$$P(y^*, z) = [y^*, x_1, x_2, \ldots, x_p = y = v, y_1, y_2, \ldots, y_r, z]$$

is a y^*-z geodesic. Thus, $x \in I[y^*, z]$, where $y^*, z \in B$. Hence $x \in I[B]$.

Therefore, B is a geodetic cover of G, contrary to our assumption that A is a geodetic basis.

Accordingly, $v \notin A$. □

Corollary 2.11 *Let G be a non-trivial connected graph and S a geodetic basis of $G \circ H$. If $v \in V(G)$, then $v \notin S$.*

Lemma 2.12 *Let G be a non-trivial connected graph of order n. If A_k is a geodetic cover of $u_k + H_{u_k}$, then $\cup_{k=1}^{n} A_k$ is a geodetic cover of $G \circ H$.*

Proof: Let A_k be a geodetic cover of $u_k + H_{u_k}$, $S = \cup_{k=1}^{n} A_k$ and $x \in V(G \circ H)$. Then $x \in V(u_k + H_{u_k}) = I_{u_k + H_{u_k}}[A_k]$ for some $u_k \in V(G)$. This implies that

$$x \in I_{u_k+H_{u_k}}[A_k] \subseteq I_{G \circ H}[A_k] \subseteq I_{G \circ H}[\cup_{k=1}^{n} A_k] = I_{G \circ H}[S]$$

by Lemma 2.7. Hence $V(G \circ H) \subseteq I_{G \circ H}[S]$. Therefore, S is a geodetic cover of $G \circ H$. □

Lemma 2.13 *Let G be a non-trivial connected graph of order n. If S is a geodetic basis of $G \circ H$, then*

$$S = \bigcup_{i=1}^{n} [S \cap V(H_{u_i})] = \bigcup_{i=1}^{n} [S \cap V(u_i + H_{u_i})] .$$

Proof: First, note that $S \cap V(H_{v_i}) \neq \emptyset$ for all i by Corollary 2.9. Now, let S be a geodetic basis of $G \circ H$. Clearly,

$$\bigcup_{i=1}^{n} [S \cap V(u_i + H_{u_i})] = S \bigcap \left[\bigcup_{i=1}^{n} V(u_i + H_{u_i}) \right] \subseteq S .$$

Now, if $x \in S$, then $x \notin V(G)$ by Corollary 2.11. It follows from Definition 2.1 that $x \in \cup_{i=1}^{n} V(H_{v_i}) \subseteq \cup_{i=1}^{n} V(u_i + H_{u_i})$. This implies that $x \in \cup_{i=1}^{n} [S \cap V(u_i + H_{u_i})]$, showing that $S = \cup_{i=1}^{n} [S \cap V(u_i + H_{u_i})]$.

Next, let $y \in S$. Then $y \notin V(G) = \cup_{i=1}^{n}\{u_i\}$ by Corollary 2.11. Thus $S \cap \{u_i\} = \emptyset$ for all $i = 1, \ldots, n$. Since $S = \cup_{i=1}^{n}[S \cap V(u_i + H_{u_i})]$, it follows that

$$S = \bigcup_{i=1}^{n}[S \bigcap (\{u_i\} \bigcup V(H_{u_i}))] = \bigcup_{i=1}^{n}[(S \bigcap \{u_i\}) \bigcup (S \bigcap V(H_{u_i}))].$$

Therefore, $S = \bigcup_{i=1}^{n}[S \bigcap V(H_{u_i})]$. □

Lemma 2.14 [8] *Every geodetic cover of a graph contains its extreme vertices.*

If H is complete and $u \in V(G)$, then every element $v \in V(H_u)$ is an extreme vertex in $V(G \circ H)$. By Lemma 2.14, we have the following.

Corollary 2.15 *Let S be a geodetic cover of $G \circ H$. If H is complete, then $V(H_u) \subseteq S$ for all $u \in V(G)$.*

Lemma 2.16 *Let G be a non-trivial connected graph and S a geodetic basis of $G \circ H$. If H is complete, then $S = \cup_{u \in V(G)} V(H_u)$.*

Proof: Suppose S is a geodetic basis of $G \circ H$. This implies that

$$S = \bigcup_{u \in V(G)} (S \cap V(H_u)) = \bigcup_{u \in V(G)} V(H_u).$$

by Lemma 2.13 and Corollary 2.15.

Theorem 2.17 *Let G be a connected graph of order n. Then*

$$g(G \circ K_m) = \begin{cases} m+1 & \text{if } G \text{ is trivial,} \\ mn & \text{if } G \text{ is non-trivial.} \end{cases}$$

Proof: Suppose G is trivial. Then $G \circ K_m = K_m + K_1 = K_{m+1}$. Thus, $g(G \circ K_m) = g(K_{m+1}) = m + 1$ by Lemma 1.1.

Suppose G is non-trivial and S is a geodetic basis of $G \circ K_m$. Since $H = K_m$, it follows that $S = \cup_{u \in V(G)} V(H_u)$ by Lemma 2.16. Therefore, $g(G \circ K_m) = n|V(H_u)| = n|V(H)| = nm$. □

Corollary 2.18 *Let n and m be two positive integers, where $n \geq 2$. Then $g(K_n \circ K_m) = nm$.*

Lemma 2.19 *Let G be a non-trivial connected graph of order n and H a non-complete graph. If S is a geodetic basis of $G \circ H$, then $S = \cup_{i=1}^{n} A_i$, where A_i is a geodetic basis of $u_i + H_{u_i}$.*

Proof: Let S be a geodetic basis of $G \circ H$. Put $A_i = S \cap V(H_{u_i})$. By Lemma 2.13,

$$S = \bigcup_{i=1}^{n} [S \cap V(u_i + H_{u_i})] .$$

Thus,

$$S = \bigcup_{i=1}^{n} [S \cap V(H_{u_i})]$$

by Lemma 2.16. Hence $S = \cup_{i=1}^{n} A_i$. Let $k \in \{1, 2, \ldots, n\}$ and $v \in V(u_k + H_{u_k})$.

Consider the following cases:

Case 1. Suppose $V(H_{u_k}) \subseteq S$. If $v \in V(H_{u_k})$, then the vertex v is in $S \cap V(H_{u_k}) = A_k \subseteq I_{u_k + H_{u_k}}[A_k]$. Now, let $v = u_k$. Since H_{u_k} is non-complete, there exist $x, y \in V(H_{u_k})$ such that $d_{G \circ H}(x, y) = 2$. Thus, $v \in I_{u_k + H_{u_k}}[x, y] \subseteq I_{u_k + H_{u_k}}[A_k]$. Hence $A_k = S \cap V(u_k + H_{u_k})$ is a geodetic cover of $u_k + H_{u_k}$.

Case 2. Suppose there exists $w \in V(H_{u_k})\setminus S$. Since S is a geodetic basis of $G \circ H$, there exist $x, y \in S$ such that $w \in I_{G\circ H}[x, y]$. It is not possible that both x and y are outside A_k because there is no path x-y containing w. Now, suppose without loss of generality that $x \in A_k$ and $y \notin A_k$. Since $w \neq x$ and $wu_k, xu_k \in E(G \circ H)$, it follows that the existence of an x-y geodesic containing w is not possible. Thus, $x, y \in A_k$. Hence $d_{G\circ H}(x, y) = 2$ and $v = u_k \in I_{u_k+H_{u_k}}[x, y] \subseteq I_{u_k+H_{u_k}}[A_k]$. Now, if $v \in V(H_{u_k})\setminus A_k$, then by following an earlier argument, we find that there exist $a, b \in A_k$ such that $v \in I_{u_k+H_{u_k}}[a, b]$. Therefore, $V(u_k + H_{u_k}) = I_{u_k+H_{u_k}}[A_k]$; that is, A_k is a geodetic cover of $u_k + H_{u_k}$.

Moreover, A_k is a geodetic basis of $u_k + H_{u_k}$ by Lemma 2.12 and the fact that S is a geodetic basis of $G \circ H$. □

Theorem 2.20 *Let G be a connected graph of order n and H a non-complete graph. Then $g(G \circ H) = n \cdot g(H + K_1)$.*

Proof: Suppose G is trivial. Then $G \circ H = H + K_1$. This implies that $g(G \circ H) = g(H + K_1)$.

Suppose G is nontrivial and let S be a geodetic basis of $G \circ H$. Then by Lemma 2.19, $S = \cup_{k=1}^{n} A_k$, where A_k is a geodetic basis of $u_k + H_{u_k}$. Thus,

$$g(G \circ H) = |S| = \sum_{k=1}^{n} |A_k| = \sum_{k=1}^{n} g(u_k + H_{u_k}) = ng(H + K_1). \qquad \square$$

Corollary 2.21 *Let n and m be two positive integers. Let G be a connected graph of order n. Then*

1. $g(G \circ P_m) = ng(F_m)$ for $m \geq 3$; and

2. $g(G \circ C_m) = ng(W_m)$ *for* $m \geq 4$.

Observe that the graph $H' + K_m$ is non-complete if and only if H' is non-complete. This notion is used in the following results.

Lemma 2.22 *Let* H *be a non-complete graph. Then for all* n,

$$g(H + K_1) = g(H + K_n). \tag{1}$$

Proof: Suppose H is non-complete. Clearly assertion (1) holds for $n = 1$. Suppose the assertion holds for $n \geq 2$; that is, $g(H + K_1) = g(H + K_r)$. Now,

$$g(H + K_{n+1}) = g((H + K_1) + K_n) = g(G + K_n)\ ,$$

where $G = H + K_1$ is non-complete.

Thus,

$$g(H + K_{n+1}) = g(G + K_1) = g(H + K_2) = g(H + K_1)\ ,$$

by the inductive hypothesis. Therefore, (1) holds for all n. □

Theorem 2.23 *Let* G *be a connected graph of order* n *and* H *a non-complete graph. If* $H = H' + K_m$ *for some graph* H', *then* $g(G \circ H) = ng(H)$.

Proof: Suppose H is a non-complete graph and for some graph H', assume that $H = H' + K_m$. By the non-completeness of H' and by Theorem 2.20, we have

$$g(G \circ H) = ng(H + K_1) = ng((H' + K_m) + K_1) = ng(H' + K_{m+1})\ .$$

This implies that

$$g(G \circ H) = ng(H' + K_1) = ng(H' + K_m)$$

by Lemma 2.22.

Therefore, $g(G \circ H) = ng(H)$. □

The following result is a direct consequence of Theorem 2.23.

Corollary 2.24 *Let n and m be two positive integers. Let G be a connected graph of order n. Then*

1. $g(G \circ F_m) = ng(F_m)$ *for* $m \geq 3$ *and*
2. $g(G \circ W_m) = ng(W_m)$ *for* $m \geq 4$.

Definition 2.25 [5] Let G be a connected graph. A subset S of $V(G)$ is a *closure absorbing set* in G if for every $x \in V(G)\backslash S$, there exist $a_x, b_x \in N(x) \cap S$ with $d_G(a_x, b_x) = 2$.

Corollary 2.26 [5] *Let G be a connected graph of order n and K_m the complete graph of order m. If G is non-complete, then*

$$g(G + K_m) = min\{|S| : S \subseteq V(G) \text{ and is closure absorbing}\}.$$

Theorem 2.27 *Let G and H be connected graphs, where G is of order n and H is non-complete. Then*

$$g(G \circ H) = n \cdot min\{|S| : S \subseteq V(H) \text{ and is closure absorbing}\}.$$

Proof: Suppose G and H are connected graphs. Then $g(G \circ H) = ng(H + K_1)$ by Theorem 2.20. Thus, for all m, $g(G \circ H) = ng(H + K_m)$ by Lemma 2.22. Hence

$$g(G \circ H) = n \cdot min\{|S| : S \subseteq V(H) \text{ and is closure absorbing}\}$$

by Corollary 2.26. □

References

[1] M. Atici, Graph operations and geodetic numbers. *Congressus Numerantium* **141** (1999) 95-110.

[2] F. Buckley, and F. Harary, *Distance in Graphs*. Addison-Wesley, Redwood City, California, 1990.

[3] G. B. Cagaanan, On geodesic convexity in graphs, Ph. D. Dissertation, MSU-IIT, 2004.

[4] G. B. Cagaanan and S. R. Canoy Jr., On the geodetic covers and geodetic bases of the composition $G[K_m]$, *Ars Combinatoria* (In press)

[5] S. R. Canoy Jr. and G. B. Cagaanan, On the geodesic and hull numbers of the sum of graphs. *Congressus Numerantium* **161**(2003) 97-104.

[6] S. R. Canoy, Jr., G. B. Cagaanan and S. V. Gervacio, Convexity, geodetic, and hull numbers of the join of graphs, *Utilitas Mathematica* (In press)

[7] G. Chartrand, F. Harary, and P. Zhang, Geodetic sets in graphs. *Discuss. Math. Graph Theory* **20**(2000) 129-138.

[8] G. Chartrand, F. Harary, and P. Zhang, On the geodetic number of a graph. *Networks* **39** (2002) 1-6.

[9] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, 3rd ed., Chapman and Hall, New York, 1996.

[10] F. Harary, *Graph Theory*. Addison-Wesley, Reading MA, 1969.

[11] F. Harary, E. Loukakis and C. Tsouros, The geodetic number of a graph. *Mathl. Comput. Modeling* **17** (1993) 89-95.