On the Topologies Induced by Some Other Special Graphs

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Abstract

The author [2] with S.R. Canoy presented results in the topologies induced by some special graphs. In this paper, some results on the topologies induced by some other special graphs are given as well as other proofs for some results in [2] and results of this study.

Keywords: topology, induced, graph, base, subbase

1 Introduction

S. Diesto and S. Gervacio [3] proved the following proposition.

Proposition 1.1 *Let $G = (X, E)$ be a simple graph. Then G induces a topology on X, denoted by $\tau(G)$, with base consisting of the form $F_G(A) = X \setminus N_G(A)$, where $N_G(A) = A \cup \{x : [x, a] \in E \text{ for some } a \in A\}$ and A ranges over all subsets of X.*

S. Gervacio and R. Guerrero [4] characterized the graphs which induce discrete and indiscrete topological spaces. S.R. Canoy and the present author

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in [1] proved results on the topologies induced by graphs by applications of the operations on graphs. In [2], they described the topologies induced by some special graphs; namely, path, fan, wheel and the star graphs.

In this paper, we will describe the topologies induced by some other special graphs; namely, the complete bipartite graphs, the empty graphs and the cycle graphs. Furthermore, we will show other proofs of some results in [2] and in this paper using the results in [1], [2] and this present paper.

Throughout this paper, all graphs are simple graphs. Moreover, if $G = (X, E)$ is a simple graph where X is the vertex set and E is the edge set of G, we denote by β_G, the base of $\tau(G)$ consisting of all sets of the form $F_G(A)$. Finally, we have $B_i = B_i^- \cup B_i^+$, where $B_i^- = \{x_j : j < i-1\}$ and $B_i^+ = \{x_j : j > i+1\}$.

2 Preliminaries

A family $\mathcal{B}$ of subsets of a nonempty set X is a base for some topology on X if and only if $X = \cup\{B : B \in \mathcal{B}\}$ and for any $B_1, B_2 \in \mathcal{B}$, if $p \in B_1 \cap B_2$ then there exists $B_p \in \beta$ such that $p \in B_p \subseteq B_1 \cap B_2$. A nonempty family S of subsets of X is a subbase if the finite intersection of members of S form a base for some topology on X.

The following results were proved in [1].

Lemma 2.1 (Key Lemma) *Let $G = (X, E)$ be a graph. Then the sets $F_G(\{a\})$, where a ranges over all elements of X, form the induced topology $\tau(G)$ of G.*

Theorem 2.2 *Let $G = (X_1, E_1)$ and $H = (X_2, E_2)$ be graphs. Then*

$$F_{G+H}(A) = \begin{cases} \varnothing & , if\ A \cap X_1 \neq \varnothing and A \cap X_2 \neq \varnothing \\ F_G(A) & , if A \subset X_1, A \neq \varnothing \\ F_H(A) & , if A \subset X_2, A \neq \varnothing \\ X_1 \cup X_2 & , if A = \varnothing \end{cases}$$

3 Results

Theorem 3.1 *Let $G = \overline{K}_m + \overline{K}_n$ (the complete bipartite graph) where y_j, $j = 1, 2, \ldots, m$, are the vertices of $\overline{K}_m$ and x_i, $i = 1, 2, \ldots, n$, are the vertices of $\overline{K}_n$. Then $\tau(G)$ has a subbase consisting of the sets $V(\overline{K}_n) \setminus \{x_i\}$ and $V(\overline{K}_m) \setminus \{y_j\}$.*

Proof: Observe that for all $i = 1, 2, \ldots, n$, we have $F_G(x_i) = V(\overline{K}_n \setminus \{x_i\}$ and for all $j = 1, 2, \ldots, m$, we have $F_G(y_j) = V(\overline{K}_m \setminus \{y_j\}$. Hence, from the Key Lemma, we have the desired result. □

Theorem 3.2 *Let $G = \overline{K}_m + \overline{K}_n$ (the complete bipartite graph) where y_j, $j = 1, 2, \ldots, m$, are the vertices of $\overline{K}_m$ and x_i, $i = 1, 2, \ldots, n$, are the vertices of $\overline{K}_n$. Then for each $i = 1, 2, \ldots, n$, $\{x_i\} \in \tau(G)$ and for all $j = 1, 2, \ldots, m$, $\{y_j\} \in \tau(G)$.*

Proof: Note that for each $i = 1, 2, \ldots, n$, $F(V(\overline{K}_n) \setminus \{x_i\}) = \{x_i\}$ and for each $j = 1, 2, \ldots, m$, $F(V(\overline{K}_m) \setminus \{y_j\}) = \{y_j\}$. Therefore, $\{x_i\} \in \tau(G)$ for all $i = 1, 2, \ldots, n$ and $\{y_j\} \in \tau(G)$ for all $j = 1, 2, \ldots, m$. □

Theorem 3.3 *Let $G = \overline{K}_n$, where $\overline{K}_n$ is the graph consisting of the isolated vertices $x_1, x_2, \ldots, x_n$ (empty graph). Then $\tau)G)$ has a subbase consisting of the sets $V(\overline{K}_n \setminus \{x_i\}$, where $i = 1, 2, \ldots, n$.*

Proof: Observe that for all $i = 1, 2, \ldots, n$, we have $F_G(x_i) = V(\overline{K}_n)\setminus\{x_i\}$. Thus, from the Key Lemma, we have the desired result. □

Theorem 3.4 *Let $G = \overline{K}_n$, where $\overline{K}_n$ is the graph consisting of the isolated vertices $x_1, x_2, \ldots, x_n$ (empty graph). Then $\{x_i\} \in \tau(G)$ for all $i = 1, 2, \ldots, n$.*

Proof: Since for each $i = 1, 2, \ldots, n$, $F(V(\overline{K}_n \setminus \{x_i\}) = \{x_i\}$, then $\{x_i\} \in \tau(G)$. □

Theorem 3.5 *Let $G = C_n$ (cycle), where $C_n = [x_1, x_2, \ldots, x_n, x_1]$ and $n \geq 3$. Then $\tau(G)$ has a subbase consisting of the sets $B_1 \setminus \{x_n\}$, $B_n \setminus \{x_1\}$ and the sets B_i's for $i \neq 1, n$.*

Proof: In view of the Key Lemma, it is enough to show that $F_G(x_i) = B_i$ for all $i \neq 1, n$ and that $F_G(x_1) = B_1 \setminus \{x_n\}$ and $F_G(x_n) = B_n \setminus \{x_1\}$. Observe that $F_G(x_1) = \{x_3, x_4, \ldots, x_{n-1}\} = B_1 \setminus x_n$, $F_G(x_n) = \{x_2, x_3, \ldots, x_{n-2}\} = B_n \setminus \{x_1\}$ and $F_G(x_i) = \{x_1, x_2, \ldots, x_{i-2}, x_{i+2}, \ldots, x_n\} = B_i$ for all $i \neq 1, n$. Thus, we have shown that $\tau(G)$ has a subbase consisting of the sets $B_1 \setminus \{x_n\}$, $B_n \setminus \{x_1\}$ and the sets B_i's for all $i \neq 1, n$. □

Theorem 3.6 *Let $G = C_n$ (cycle), where $C_n = [x_1, x_2, \ldots, x_n, x_1]$ and $n \geq 3$. Then $\{x_i\} \in \tau(G)$ for each $i = 1, 2, \ldots, n$.*

Proof: Note that $F_G(B_1 \setminus \{x_n\}) = \{x_1\}$, $F_G(B_n \setminus \{x_1\}) = \{x_n\}$, and $F_G(B_i) = \{x_i\}$ for all $i \neq 1, n$. □

Thus, for all $i = 1, 2, \ldots, n$, $\{x_i\} \in \tau(G)$.

4 Alternative Proofs

The following theorems were presented in [2]. We will give alternative proofs for these theorems using results in [1] and the previous section.

Theorem 4.1 *Let $G = W_n = K_1 + C_n$ (wheel), where $K_1 = [v_0]$, $C_n = [x_1, x_2, \ldots, x_n, x_1]$ and $n \geq 3$. Then $\tau(G)$ has a subbase consisting of the sets $B_1 \setminus \{x_n\}$, $B_n \setminus \{x_1\}$, and the sets B_i's for $i \neq 1, n$.*

Proof: In view of our Key Lemma, it suffices to show that $F_G(x_i) = B_1 \setminus \{x_n\}$ or $B_n \setminus \{x_1\}$ or B_i. From Theorem 2.2 and Theorem 3.5 we have

$$F_G(x_1) = F_{C_n}(x_1) = B_1 \setminus \{x_n\},$$

$$F_G(x_n) = F_{C_n}(x_n) = B_n \setminus \{x_1\}; and,$$

$$F_G(x_i) = F_{C_n}(x_i) = B_i, \text{ where } i \neq 1, n.$$

Hence, $\tau(G)$ has a subbase consisting of the sets $B_1 \setminus \{x_n\}$, $B_n \setminus \{x_1\}$ and the sets B_i's for $i \neq 1, n$. □

Theorem 4.2 *Let $G = W_n = K_1 + C_n$ (wheel), where $K_1 = [v_0]$, $C_n = [x_1, x_2, \ldots, x_n, x_1]$ and $n \geq 3$. Then $\{x_i\} \in \tau(G)$ for each $i = 1, 2, \ldots, n$ but $\{v_0\}$ is not in $\tau(G)$.*

Proof: Observe that $\{v_0\}$ is not in $\tau(G)$ since for all $i = 1, 2, \ldots, n$, $[v_0, x_i] \in E$ and so, $\{v_0\} \neq F_G(A)$ for any $A \subseteq X$. Therefore, $\{v_0\}$ is not in $\tau(G)$.

Now, from Theorems 2.2 and 3.6,

$$F_G(B_1 \setminus x_n) = F_{C_n}(B_1 \setminus x_n) = x_1,$$

$$F_G(B_n \setminus x_1) = F_{C_n}(B_n \setminus x_1) = x_n; and,$$

$$F_G(B_i) = F_{C_n}(B_i) = x_i \text{ where } i \neq 1, n.$$

Therefore, $\{x_i\} \in \tau(G)$ for each $i = 1, 2, \ldots, n$. □

Theorem 4.3 *Let $G = K_{1,n} = K_1 + \overline{K}_n$ (star), where $K_1 = [v_0]$ and $\overline{K}_n$ is the empty graph. Then $\tau(G)$ has a subbase consisting of the sets $V(\overline{K}_n) \setminus \{x_i\}$ where $i = 1, 2, \ldots, n$.*

Proof: In view of the Key Lemma, it suffice to show that $F_G(x_i) = V(\overline{K}_n \setminus \{x_i\}$ for all $i = 1, 2, \ldots, n$ and for all $j = 1, 2, \ldots, m$, $F_G(y_j) = V(\overline{K}_m) \setminus \{y_j\}$.

From Theorem 2.2, $F_G(x_i) = F_{\overline{K}_n}(x_i)$. Now, from Theorem 3.3, $F_{\overline{K}_n}(x_i) = V(\overline{K}_n) \setminus \{x_i\}$.

Therefore, $\tau(G)$ has a subbase consisting of the sets $V(\overline{K}_n) \setminus \{x_i\}$ for each $i = 1, 2, \ldots, n$. □

Theorem 4.4 *Let $G = K_{1,n} = K_1 + \overline{K}_n$ (star), where $K_1 = [v_0]$ and $\overline{K}_n$ is the empty graph. Then $\{x_i\} \in \tau(G)$ for all $i = 1, 2, \ldots, n$ but $\{v_0\}$ is not in $\tau(G)$.*

Proof: From Theorem 2.2, $F_G(B_i) = F_{\overline{K}_n}(B_i)$. It is clear from Theorem 3.4 that for each $i = 1, 2, \ldots, n$, $F_{\overline{K}_n}(B_i) = \{x_i\}$. Thus, $\{x_i\} \in \tau(G)$.

Furthermore, $\{v_0\}$ is not in $\tau(G)$ since for all $i = 1, 2, \ldots, n$, $[v_0, x_i] \in X$ and thus $\{v_0\} \neq F_G(A)$ for any $A \subseteq E$. □

Hence we have proved the theorem.

At this point we will present another way of proving Theorems 3.1 and 3.2.

Another proof of Theorem 3.1: First, we prove Theorem 3.1.

In view of the Key Lemma, it is enough to show that $F_{\overline{K}_n}(x_i) = V(\overline{K}_n) \setminus \{x_i\}$ for all $i = 1, 2, \ldots, n$ and $F_{\overline{K}_m}(y_j) = V(\overline{K}_m) \setminus \{y_j\}$ for all $i = 1, 2, \ldots, n$

From Theorem 2.2, $F_G(B_i) = F_{\overline{K}_n}(B_i)$ if $B_i \subseteq V(\overline{K}_n)$ and $F_G(B_i) = F_{\overline{K}_m}(B_i)$ if $B_i \subseteq V(\overline{K}_m)$. Thus, from Theorem 3.3, for all $i = 1, 2, \ldots, n$ $F_{\overline{K}_n}(x_i) = V(\overline{K}_n) \setminus \{x_i\}$ and for all $j = 1, 2, \ldots, m$ $F_{\overline{K}_m}(y_j) = V(\overline{K}_m) \setminus \{y_j\}$. This completes the proof of the theorem. □

Next, we shall prove Theorem 3.2. It is obvious from Theorem 2.2 that $F_G(B_i) = F_{\overline{K}_n}(B_i)$ if $B_i \subseteq V(\overline{K}_n)$ and $F_G(B_i) = F_{\overline{K}_m}(B_i)$ if $B_i \subseteq V(\overline{K}_m)$. Thus, from Theorem 3.4, for all $i = 1, 2, \ldots, n$, $F_{\overline{K}_n}(V(\overline{K}_n) \setminus \{x_i\}) = \{x_i\}$ and for all $j = 1, 2, \ldots, m$, $F_{\overline{K}_m}(V(\overline{K}_m) \setminus \{y_j\}) = \{y_j\}$. □

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