A Laplace Transform Technique for Evaluating Infinite Series

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Abstract

In one of the articles in the *Mathematics Magazine Vol. 76, No. 5*, Costas Efthimiou (1999), the author of the article "Finding Exact Values for Infinite Series", showed how the Laplace transform can be used as a tool for evaluating infinite series. Efthimiou's article is the main reference of James P. Lesko and Wendy D. Smith in their article entitled "A Laplace Transform Technique in Evaluating Infinite Series". This study is an exposition as well as an extension of Lesko and Smith's article. Moreover, it is a demonstration on how the Laplace transform can be used as a tool to find closed form expressions for certain infinite series. There were cases when the summands of an infinite series can be realized as a Laplace transform integral which was then used as a tool in evaluating these series. That is, if $\sum_{n=1}^{\infty} u_n$ is an infinite series whose summand u_n, can be realized as a Laplace transform integral $u_n = \int_0^{\infty} e^{-nx} F(x)dx$ then,

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nx} F(x)dx = \int_0^{\infty} F(x)\left(\frac{e^{-x}}{1-e^{-x}}\right)dx.$$

Justification on these manipulations is further discussed in this paper. Moreover, the purpose of this article is to explain this technique, and to illustrate with several examples.

Keywords: Laplace transform, infinite series

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Introduction

Applying the Laplace transform technique in infinite series is a new technique to be explored. Specifically, the intention of this paper is to find the sum of the infinite series with the help of the Laplace operator.

The concept of this study was introduced by Costas Efthimiou in his article entitled **"Finding Exact Values for Infinite Sums"**. Further studies regarding the said concept were analyzed and examined by James P. Lesko and Wendy D. Smith in their article entitled **"A Laplace Transform Technique in Evaluating Infinite Series"**. Compared to Efthimiou's technique, a better development of the concept was established by Lesko and Smith where they illustrated a more general application of the technique.

Efthimiou found expressions for series of the form

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)}$$

where $a, b \in Z^{-}$. He applies the same methods to the sum of the series of the form

$$\sum_{n=1}^{\infty} \frac{Q(n)}{P(n)}$$

where P and Q are polynomials with $deg(P) - deg(Q) = 2$ and P factors completely into linear factors with no roots in Z^{+}, that is, $(n-r_1)(n-r_2)\cdots(n-r_n)$ such that $r_i \notin Z^{+}$.

Efthimiou's technique applies when the summation

$$\sum_{n=1}^{\infty} u_n$$

is an infinite series whose summands u_n can be realized as a Laplace transform integral

$$u_n = \int_0^{\infty} e^{-nx} F(x)dx.$$

In such a case, what appeared to be a sum of numbers is now written as a sum of integrals. That is,

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nx} F(x)dx$$

$$= \int_0^{\infty} f(x) \sum_{n=1}^{\infty} e^{-nx}\, dx \qquad (1)$$

$$= \int_0^{\infty} f(x) \left(\frac{e^{-x}}{1-e^{-x}}\right) dx.$$

If the integral can be easily evaluated, then the sum is solvable.

Now, Efthimiou's technique can be generalized to series of the form

$$\sum_{n=1}^{\infty} u_n v_n$$

where it is convenient to write only u_n as a Laplace transform integral. Again, series can be written as a sum of integrals, but with the factor v_n. That is,

$$\sum_{n=1}^{\infty} u_n v_n = \sum_{n=1}^{\infty} v_n \int_0^{\infty} e^{-nx} F(x)dx.$$

Objectives

The objectives of this paper are the following:

1. To offer a new technique, the Laplace transform technique in evaluating infinite series.
2. To show how the well-known properties of real analysis can be applied in the process of evaluating the infinite series.
3. To prove some well-known identities regarding infinite series.

Preliminaries

Definition 3.1 [9] For a function $F(x)$ defined on $0 \le x < \infty$, its *Laplace transform* is denoted as $f(n)$ obtained by the integral

$$L\{F(x)\} = \int_0^\infty e^{-nx} F(x)dx = f(n)$$

where $n \in N$ and L is the *Laplace transform operator.*

Definition 3.2 [9] If the Laplace transform of $F(x)$ is $f(n)$, then we say that the *inverse Laplace transform* of $f(n)$ is $F(x)$ which is denoted by

$$L^{-1}\{f(n)\} = F(x)$$

where L^{-1} is called the *inverse Laplace transform operator.*

Theorem 3.3 [2] The Laplace transform integral of $\frac{1}{A^L}$ for $A > 0$ and $L \ge 1$ is given by

$$\frac{1}{A^L} = \frac{1}{(L-1)!}\int_0^\infty x^{L-1} e^{-Ax} dx.$$

Theorem 3.4 *(Abel's Theorem)* [1]

Let $\sum_{n=0}^{\infty} a_n x_n$ have radius R, and let it also converge for $x = R$ (resp. $x = -R$). Then, it is uniformly convergent on the interval $0 \le x \le R$ (resp. the interval $-R \le x \le 0$).

Theorem 3.5 *(Dominated Convergence Theorem)* [7]

Let v be a nondecreasing function defined on $\mathbb{R}$ and suppose that $\{f_n\}$ is a pointwise convergent sequence for function in $\vartheta(v)$ such that $|f_n| \le g$ for all n and some function $g \in \vartheta(v)$ and satisfies

$$\lim_{n\to\infty} \int_R f_n dv = \int_R (\lim_{n\to\infty} f_n)\, dv.$$

Theorem 3.6 *(Monotone Convergence Theorem)* [16]

If for all natural numbers j and k, $a_{j,k}$ is a non-negative real number and $a_{j,k} \le a_{j+1,k}$, then

$$\sum_k \lim_{j\to\infty} a_{j,k} = \lim_{j\to\infty} \sum_k a_{j,k}.$$

Results

Lemma 4.1 For $b > a > -1$,

$$\frac{1}{(n+a)(n+b)} = \int_0^\infty e^{-nx}\left(\frac{e^{-ax} - e^{-bx}}{b-a}\right)dx.$$

Proof. By partial fractions,

$$\frac{1}{(n+a)(n+b)} = \frac{1}{b-a}\left(\frac{1}{n+a} - \frac{1}{n+b}\right).$$

Applying the inverse Laplace transform,

$$L^{-1}\left\{\frac{1}{(n+a)(n+b)}\right\}$$
$$= L^{-1}\left\{\frac{1}{b-a}\left(\frac{1}{n+a} - \frac{1}{n+b}\right)\right\}. \quad (2)$$

Then,

$$L^{-1}\left\{\frac{1}{b-a}\left(\frac{1}{n+a} - \frac{1}{n+b}\right)\right\} = \frac{1}{b-a}L^{-1}\left\{\frac{1}{n+a} - \frac{1}{n+b}\right\}.$$

This means that,

$$\frac{1}{b-a}L^{-1}\left\{\frac{1}{n+a} - \frac{1}{n+b}\right\} = \frac{1}{b-a}\left(L^{-1}\left\{\frac{1}{n+a}\right\} - L^{-1}\left\{\frac{1}{n+b}\right\}\right).$$

Claim: $L^{-1}\left\{\frac{1}{n+a}\right\} = e^{-ax}$ and $L^{-1}\left\{\frac{1}{n+b}\right\} = e^{-bx}$

But note that

$$\frac{1}{n+1} = \int_0^\infty e^{-(n+a)x}dx.$$

Hence,

$$\int_0^\infty e^{-(n+a)x}dx = \int_0^\infty e^{-nx}(e^{-ax})dx.$$

In addition, by definition,

$$\int_0^\infty e^{-nx}(e^{-ax})dx = L\{e^{-ax}\}$$

so that

$$\frac{1}{n+a} = L\{e^{-ax}\}$$

and

$$L^{-1}\left\{\frac{1}{n+a}\right\} = e^{-ax}.$$

Similarly, it can be shown that

$$L^{-1}\left\{\frac{1}{n+b}\right\} = e^{-bx}.$$

This proves the claim. Thus from the claim above,

$$\frac{1}{b-a}\left(L^{-1}\left\{\frac{1}{n+a}\right\} - L^{-1}\left\{\frac{1}{n+b}\right\}\right) = \frac{1}{b-a}(e^{-ax} - e^{-bx}).$$

But,

$$L^{-1}\left\{\frac{1}{(n+a)(n+b)}\right\} = \frac{1}{b-a}(e^{-ax} - e^{-bx}).$$

Hence,

$$\frac{1}{(n+a)(n+b)} = L\left\{\frac{e^{-ax} - e^{-bx}}{b-a}\right\}.$$

Therefore,

$$\frac{1}{(n+a)(n+b)} = \int_0^\infty e^{-nx}\left(\frac{e^{-ax} - e^{-bx}}{b-a}\right)dx.$$

■

Theorem 4.2 For $b > a > -1$, the infinite series

$$\sum_{n=1}^{\infty}\frac{1}{(n+a)(n+b)} = \frac{1}{b-a}\int_0^1 \frac{u^a - u^b}{1-u}\,du$$

Proof. Let $b > a > -1$. By Lemma 4.1,

$$\sum_{n=1}^{\infty}\frac{1}{(n+a)(n+b)} = \sum_{n=1}^{\infty}\int_0^\infty e^{-nx}\left(\frac{e^{-ax} - e^{-bx}}{b-a}\right)dx.$$

For $j, n \in N$, let

$$W_{j,n} = \lim_{j\to\infty}\int_0^j e^{-nx}\frac{e^{-ax}-e^{-bx}}{b-a}\,dx.$$

Since $b > a$, thus $e^b > e^a$ implying that $e^{-a} > e^{-b}$ or $e^{-ax} > e^{-bx}$ for $0 < x < \infty$. Also, notice that the factors $b-a$, e^{-nx}, $e^{-ax}-e^{-bx}$ are all nonnegative for $0 < x < \infty$. Thus $W_{j,n} \geq 0$.

Next, show that

$W_{j,n} \leq W_{j+1,n}$.

Since

$$W_{j,n} = \frac{1}{b-a}\lim_{j\to\infty}\int_0^j \left[e^{-x(n+a)} - e^{-x(n+b)}\right]dx,$$

then

$$W_{j+1,n} = \frac{1}{b-a}\lim_{j\to\infty}\int_0^{j+1} \left[e^{-x(n+a)} - e^{-x(n+b)}\right]dx.$$

Notice that the integral $W_{j,n}$ can be interpreted as the difference of the areas of the region bounded by the curves $e^{-x(n+a)}$, $e^{-x(n+b)}$ and the positive x –axis. Likewise the integral $W_{j+1,n}$ can also be considered similarly. Thus, the area of the region represented by the integral in $W_{j,n}$ is less than the area of the region represented by the integral in $W_{j+1,n}$. Hence, for all $j, n \in N$,

$W_{j+1,n} \geq W_{j,n}$.

This shows that $W_{j,n}$ is monotonic increasing. In this case, apply the Monotone Convergence Theorem to switch the sum and integral and obtain

$$\sum_{n=1}^{\infty}\int_0^{\infty} e^{-nx}\left(\frac{e^{-ax}-e^{-bx}}{b-a}\right)dx = \int_0^{\infty}\sum_{n=1}^{\infty} e^{-nx}\left(\frac{e^{-ax}-e^{-bx}}{b-a}\right)dx$$
$$= \int_0^{\infty}\left(\frac{e^{-ax}-e^{-bx}}{b-a}\right)\sum_{n=1}^{\infty} e^{-nx}\,dx$$

Claim: $\sum_{n=1}^{\infty} e^{-nx} = \frac{e^{-x}}{1-e^{-x}}$

By manipulation,

$$\sum_{n=1}^{\infty} e^{-nx} = \sum_{n=1}^{\infty} e^{-nx+x-x}$$

$$=\sum_{n=1}^{\infty} e^{-x(n-1)-x}$$

$$=\sum_{n=1}^{\infty}(e^{-x})(e^{-x})^{n-1}.$$

This is a geometric series with $a = e^{-x}$ and $r = e^{-x}$. Hence, we have

$$\sum_{n=1}^{\infty} e^{-nx} = \frac{e^{-x}}{1-e^{-x}}.$$

This completes the claim. Now,

$$\int_0^{\infty}\left(\frac{e^{-ax}-e^{-bx}}{b-a}\right)\sum_{n=1}^{\infty} e^{-nx}dx = \frac{1}{b-a}\int_0^{\infty}(e^{-ax}-e^{-bx})\left(\frac{e^{-x}}{1-e^{-x}}\right)dx.$$

Let $u = e^{-x}$. Thus, $du = -e^{-x}dx$ which implies that $-du = e^{-x}dx$. As $x \to \infty$, $u \to 0$ and at $x = 0$, $u = 1$. Hence,

$$\frac{1}{b-a}\int_0^{\infty}(e^{-ax}-e^{-bx})\left(\frac{e^{-x}}{1-e^{-x}}\right)dx = \frac{1}{b-a}\int_0^{1}(u^a-u^b)\left(\frac{-du}{1-u}\right)$$

$$=\frac{1}{b-a}\int_0^{1}\frac{u^a-u^b}{1-u}du.$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{(n+a)(n+b)} = \frac{1}{b-a}\int_0^{1}\frac{u^a-u^b}{1-u}du.$$

■

Lemma 4.3 For $a > 0, b \geq 0$, the infinite series

$$\frac{1}{an+b} = \int_0^{\infty}\frac{e^{-nx}\left(e^{\frac{-b}{a}x}\right)}{a}dx.$$

Proof. By partial fractions,

$$\frac{1}{an+b} = \frac{1}{a}\left(\frac{1}{n+\frac{b}{a}}\right).$$

Applying the inverse Laplace transform,

$$L^{-1}\left\{\frac{1}{an+b}\right\} = L^{-1}\left\{\frac{1}{a}\left(\frac{1}{n+\frac{b}{a}}\right)\right\}.$$

But,

$$L^{-1}\left\{\frac{1}{a}\left(\frac{1}{n+\frac{b}{a}}\right)\right\} = \frac{1}{a}L^{-1}\left\{\frac{1}{n+\frac{b}{a}}\right\}.$$

Hence,

$$L^{-1}\left\{\frac{1}{an+b}\right\} = \frac{1}{a}\left(e^{\frac{-b}{a}x}\right).$$

Moreover,

$$\frac{1}{an+b} = L\left\{\frac{1}{a}\left(e^{\frac{-b}{a}x}\right)\right\}.$$

Thus,

$$L\left\{\frac{1}{a}\left(e^{\frac{-b}{a}x}\right)\right\} = \int_0^\infty \frac{e^{-nx}\left(e^{\frac{-b}{a}x}\right)}{a}\,dx.$$

Therefore,

$$\frac{1}{an+b} = \int_0^\infty \frac{e^{-nx}\left(e^{\frac{-b}{a}x}\right)}{a}\,dx.$$

■

Theorem 4.4 For $a > 0$ and $b \geq 0$, the infinite series

$$\sum_{n=1}^{\infty}\frac{1}{an+b} = \frac{1}{a}\int_0^1 \frac{u^{\frac{b}{a}}}{1-u}\,du.$$

Proof. Suppose that $a > 0$ and $b \geq 0$. Then

$$\sum_{n=1}^{\infty}\frac{1}{an+b} = \sum_{n=1}^{\infty}\int_0^\infty e^{-nx}\left(\frac{1}{a}e^{\frac{-b}{a}x}\right)dx.$$

By the Monotone Convergence Theorem,

$$\sum_{n=1}^{\infty}\int_0^\infty e^{-nx}\left(\frac{1}{a}e^{\frac{-b}{a}x}\right)dx = \int_0^\infty \sum_{n=1}^{\infty} e^{-nx}\left(\frac{1}{a}e^{\frac{-b}{a}x}\right)dx.$$

Now,

$$\int_0^\infty \sum_{n=1}^\infty e^{-nx}\left(\frac{1}{a}e^{\frac{-b}{a}x}\right)dx = \frac{1}{a}\int_0^\infty e^{\frac{-b}{a}x}\sum_{n=1}^\infty e^{-nx}dx.$$

By the claim of Theorem 4.2,

$$\frac{1}{a}\int_0^\infty e^{\frac{-b}{a}x}\sum_{n=1}^\infty e^{-nx}dx = \frac{1}{a}\int_0^\infty e^{\frac{-b}{a}x}\left(\frac{e^{-x}}{1-e^{-x}}\right)dx.$$

Suppose that $u = e^{-x}$, then $du = -e^{-x}dx$ which implies that $-du = e^{-x}dx$. At $x \to \infty, u \to 0$ and at $x = 0, u = 1$. Now

$$\frac{1}{a}\int_0^\infty e^{\frac{-b}{a}x}\left(\frac{e^{-x}}{1-e^{-x}}\right)dx = \frac{1}{a}\int_1^0 u^{\frac{b}{a}}\left(\frac{-du}{1-u}\right)$$

$$= \frac{1}{a}\int_0^1 \frac{u^{\frac{b}{a}}}{1-u}du.$$

Therefore,

$$\sum_{n=1}^\infty \frac{1}{an+b} = \frac{1}{a}\int_0^1 \frac{u^{\frac{b}{a}}}{1-u}du.$$

■

Theorem 4.5 For $a > 0$, $b \geq 0$, and $r \in [-1,1)$, the infinite series

$$\sum_{n=1}^\infty \frac{r^n}{an+b} = \frac{1}{a}\int_0^1 \frac{ru^{\frac{b}{a}}}{1-ru}du.$$

Proof. As long as $r \neq -1$, the partial sums of

$$\sum_{n=1}^\infty \frac{r^n e^{-nx}\left(e^{\frac{-b}{a}x}\right)}{a}$$

are dominated above by

$$\sum_{n=1}^{\infty}\left|r^n e^{-nx}\left(\frac{1}{a}e^{\frac{-b}{a}x}\right)\right|.$$

But,

$$\sum_{n=1}^{\infty}\left|r^n e^{-nx}\left(\frac{1}{a}e^{\frac{-b}{a}x}\right)\right| = \left|\frac{1}{a}e^{\frac{-b}{a}x}\right|\left|\sum_{n=1}^{\infty} r^n e^{-nx}\right|.$$

Since $\frac{1}{a}e^{\frac{-b}{a}x} > 0$, then $\left|\frac{1}{a}e^{\frac{-b}{a}x}\right| = \frac{1}{a}e^{\frac{-b}{a}x}$. By manipulation,

$$\begin{aligned}\left|\frac{1}{a}e^{\frac{-b}{a}x}\right|\left|\sum_{n=1}^{\infty} r^n e^{-nx}\right| &= \frac{1}{a}e^{\frac{-b}{a}x}\sum_{n=1}^{\infty}|r^n e^{-nx}|\\ &= \frac{1}{a}e^{\frac{-b}{a}x}\sum_{n=1}^{\infty}|r^n||e^{-nx}|\\ &= \frac{1}{a}e^{\frac{-b}{a}x}\sum_{n=1}^{\infty}(|r|)^n(|e^{-x}|)^n\\ &= \frac{1}{a}e^{\frac{-b}{a}x}\sum_{n=1}^{\infty}(|r|)^n(e^{-x})^n\\ &= \frac{1}{a}e^{\frac{-b}{a}x}\sum_{n=1}^{\infty}(|r|e^{-x})^n\\ &= \frac{1}{a}e^{\frac{-b}{a}x}\left(\frac{|r|e^{-x}}{1-|r|e^{-x}}\right).\end{aligned}$$

with the condition that $|r|e^{-x} < 1$.

Claim: $\int_0^{\infty}\frac{|r|e^{-x}}{1-|r|e^{-x}}dx < \infty$

From the given integral $\int_0^{\infty}\frac{|r|e^{-x}}{1-|r|e^{-x}}dx$, let $u = 1-|r|e^{-x}$. Hence, $du = |r|e^{-x}dx$. As $x \to \infty, u \to 1$ and at $x = 0, u = 1-|r|$. Now,

$$\begin{aligned}\int_0^{\infty}\frac{|r|e^{-x}}{1-|r|e^{-x}}dx &= \int_{1-|r|}^{1}\frac{du}{r}\\ &= \ln u\ |_{1-|r|}^{1}\\ &= \ln 1 - \ln(1-|r|)\\ &= -\ln(1-|r|)\end{aligned}$$

$$= \ln\left(\frac{1}{1-|r|}\right).$$

Note that as long as $r \in [-1,1)$, $\int_0^\infty \frac{|r|e^{-x}}{1-|r|e^{-x}} dx$ is finite. Thus

$$\int_0^\infty \frac{1}{a} e^{\frac{-b}{a}x}\left(\frac{|r|e^{-x}}{1-|r|e^{-x}}\right)dx \le \frac{1}{a}\int_0^\infty \left(\frac{|r|e^{-x}}{1-|r|e^{-x}}\right)dx < \infty.$$

Therefore,

$$\frac{1}{an+b} = \int_0^\infty e^{-nx}\left(\frac{1}{a}e^{\frac{-b}{a}x}\right)dx.$$

Then

$$\sum_{n=1}^{\infty} \frac{r^n}{an+b} = \sum_{n=1}^{\infty} \int_0^\infty r^n e^{-nx}\left(\frac{1}{a}e^{\frac{-b}{a}x}\right)dx.$$

We will show f_k is nondecreasing. Let

$$f_k = \sum_{n=1}^{k} r^n e^{-nx}\left(\frac{1}{a}e^{\frac{-b}{a}x}\right) \text{ and } f_{k+1} = \sum_{n=1}^{k+1} r^n e^{-nx}\left(\frac{1}{a}e^{\frac{-b}{a}x}\right)$$

Since $f_k = \frac{1}{a}e^{\frac{-b}{a}x}\sum_{n=1}^{k} r^n e^{-nx}$, $\sum_{n=1}^{k} r^n e^{-nx}$ is a geometric progression having the S_k partial sum equal to

$$S_k = \frac{re^{-x}(1-(re^{-x})^k)}{1-re^{-x}},$$

if $re^{-x} \neq 1$. It follows that

$$S_{k+1} = \frac{re^{-x}(1-(re^{-x})^{k+1})}{1-re^{-x}}.$$

Comparing S_k and S_{k+1}, to show that $S_k \le S_{k+1}$, suppose on the contrary that $S_k > S_{k+1}$. Use the terms inside the parenthesis of the two proceeding equations. Now,

$$S_k > S_{k+1}$$

$$1-(re^{-x})^k > 1-(re^{-x})^{k+1}$$

$$(re^{-x})^{k+1} > (re^{-x})^k$$

$$re^{-x} > 1.$$

But, $0 < e^x < r$ with $-1 < r < 1$. Thus, this is a contradiction. Therefore, f_k is nondecreasing. Next, use the Dominated Convergence Theorem to obtain

$$\sum_{n=1}^{\infty}\int_0^{\infty} r^n e^{-nx}\left(\frac{1}{a}e^{\frac{-b}{a}x}\right)dx = \int_0^{\infty}\sum_{n=1}^{\infty} r^n e^{-nx}\left(\frac{1}{a}e^{\frac{-b}{a}x}\right)dx$$

$$= \int_0^{\infty}\frac{1}{a}e^{\frac{-b}{a}x}\sum_{n=1}^{\infty} r^n e^{-nx}dx$$

$$= \frac{1}{a}\int_0^{\infty} e^{\frac{-b}{a}x}\left(\frac{re^{-x}}{1-re^{-x}}\right)dx.$$

Assume that $u = e^{-x}$, then $du = -e^{-x}dx$ which implies that $-du = e^{-x}dx$. As $x \to \infty, u \to 0$ and at $x = 0, u = 1$. Now,

$$\frac{1}{a}\int_0^{\infty} e^{\frac{-b}{a}x}\left(\frac{re^{-x}}{1-re^{-x}}\right)dx = \frac{1}{a}\int_1^{0} u^{\frac{b}{a}}\left(\frac{r(-du)}{1-ru}\right)$$

$$= \frac{1}{a}\int_0^{1}\frac{ru^{\frac{b}{a}}}{1-ru}du.$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{r^n}{an+b} = \frac{1}{a}\int_0^{1}\frac{ru^{\frac{b}{a}}}{1-ru}du.$$

If $r = -1$, the right hand side of the equation exists. Then this series is still convergent and by Abel's theorem, it is uniformly convergent on the interval
$-1 \le r <$
1.

Theorem 4.6 For $b > a > -1$ and $r \in [-1,1]$

$$\sum_{n=1}^{\infty}\frac{r^n}{(n+a)(n+b)} = \frac{r}{b-a}\int_0^{1}\frac{u^a - u^b}{1-ru}du.$$

Proof. By Theorem 4.1,

$$\frac{1}{(n+a)(n+b)} = \int_0^\infty e^{-nx}\left(\frac{e^{-ax} - e^{-bx}}{b-a}\right)dx.$$

As long as $r \neq -1$, the partial sums

$$\sum_{n=1}^{\infty} r^n e^{-nx}\left(\frac{e^{-ax} - e^{-bx}}{b-a}\right)$$

are dominated above by

$$\sum_{n=1}^{\infty}\left|r^n e^{-nx}\left(\frac{e^{-ax} - e^{-bx}}{b-a}\right)\right|.$$

But

$$\sum_{n=1}^{\infty}\left|r^n e^{-nx}\left(\frac{e^{-ax} - e^{-bx}}{b-a}\right)\right| = \frac{e^{-ax} - e^{-bx}}{b-a}\sum_{n=1}^{\infty}|r^n e^{-nx}|$$

$$= \frac{e^{-ax} - e^{-bx}}{b-a}\left(\frac{|r|e^{-x}}{1-|r|e^{-x}}\right),$$

where

$$\int_0^\infty \frac{1}{b-a} e^{-ax} - e^{-bx}\left(\frac{|r|e^{-x}}{1-|r|e^{-x}}\right)dx \le \frac{1}{b-a}\int_0^\infty \left(\frac{|r|e^{-x}}{1-|r|e^{-x}}\right)dx < \infty.$$

Using the same argument as the proof of the preceding theorem, we can use the Dominated Convergence Theorem to switch the sum and the integral to obtain

$$\sum_{n=1}^{\infty}\frac{r^n}{(n+a)(n+b)} = \sum_{n=1}^{\infty} r^n \int_0^\infty e^{-nx}\left(\frac{e^{-ax} - e^{-bx}}{b-a}\right)dx$$

$$= \int_0^\infty \left[\left(\frac{e^{-ax} - e^{-bx}}{b-a}\right)\sum_{n=1}^{\infty} r^n e^{-nx}\right]dx$$

$$= \frac{1}{b-a}\int_0^\infty (e^{-ax} - e^{-bx})\left(\frac{re^{-x}}{1-re^{-x}}\right)dx.$$

Suppose that $u = e^{-x}$, then $du = -e^{-x}dx$ which implies that $-du = e^{-x}dx$. At $x \to \infty$, $u \to 0$ and at $x = 0$, $u = 1$. Now,

$$\frac{1}{b-a}\int_0^{\infty}(e^{-ax}-e^{-bx})\left(\frac{re^{-x}}{1-re^{-x}}\right)dx=\frac{1}{b-a}\int_1^0(u^a-u^b)\left(\frac{r(-du)}{1-ru}\right)$$
$$=\frac{r}{b-a}\int_0^1\frac{u^a-u^b}{1-ru}\,du.$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{r^n}{(n+a)(n+b)}=\frac{r}{b-a}\int_0^1\frac{u^a-u^b}{1-ru}\,du.$$

Again, by Abel's theorem,
$-1\le r\le 1.$ ■

Lemma 4.7 Assume that $I=N^*=1,2,3,\cdots,x\in R$ and $t>0$. Then

$$\sum_{n=1}^{\infty}\cos(nx)e^{-nt}=\frac{e^{-t}(\cos x-e^{-t})}{1-2e^{-t}\cos x+e^{-2t}}$$

and

$$\sum_{n=1}^{\infty}\sin(nx)\,e^{-nt}=\frac{e^{-t}\sin x}{1-2e^{-t}\cos x+e^{-2t}}.$$

Proof. Suppose

$$C=\sum_{n=1}^{\infty}\cos(nx)e^{-nt}$$

and

$$S=\sum_{n=1}^{\infty}\sin(nx)e^{-nt}.$$

Performing the summations using complex notations,

$$C+iS=\sum_{n=1}^{\infty}\cos(nx)e^{-nt}+i\sum_{n=1}^{\infty}\sin(nx)e^{-nt}$$
$$=\sum_{n=1}^{\infty}(\cos(nx)e^{-nt}+i\sin(nx)e^{-nt})$$

$$= \sum_{n=1}^{\infty} e^{inx} e^{-nt}$$

by the Euler's identity. But

$$\sum_{n=1}^{\infty} e^{inx} e^{-nt} = \sum_{n=1}^{\infty} (e^{ix-t})^n$$

$$= \frac{e^{ix-t}}{1 - e^{-ix-t}}$$

$$= \frac{e^{-t} e^{ix}}{1 - e^{-t} e^{ix}}.$$

Again, by Euler's identity,

$$\frac{e^{-t} e^{ix}}{1 - e^{-t} e^{ix}} = \frac{e^{-t}(\cos x + i \sin x)}{1 - e^{-t}(\cos x + i \sin x)}$$

$$= \frac{e^{-t} \cos x + i e^{-t} \sin x}{(1 - e^{-t} \cos x) - i(e^{-t} \sin x)} \cdot \frac{(1 - e^{-t} \cos x) + i(e^{-t} \sin x)}{(1 - e^{-t} \cos x) + i(e^{-t} \sin x)}$$

$$= \frac{e^{-t} \cos x - e^{-2t}(\cos^2 x + \sin^2 x) + i e^{-t} \sin x}{1 - 2e^{-t} \cos x + e^{-t}(\cos^2 x + \sin^2 x)}$$

$$= \frac{e^{-t} \cos x - e^{-2t} + i e^{-t} \sin x}{1 - 2e^{-t} \cos x + e^{-2t}}$$

$$= \frac{e^{-t} \cos x - e^{-2t}}{1 - 2e^{-t} \cos x + e^{-2t}} + i \frac{e^{-t} \sin x}{1 - 2e^{-t} \cos x + e^{-2t}}.$$

Hence,

$$C = \frac{e^{-t} \cos x - e^{-2t}}{1 - 2e^{-t} \cos x + e^{-2t}}$$

and

$$S = \frac{e^{-t} \sin x}{1 - 2e^{-t} \cos x + e^{-2t}}.$$

But

$$C = \sum_{n=1}^{\infty} \cos(nx)e^{-nt}$$

and

$$S = \sum_{n=1}^{\infty} \sin(nx)e^{-nt}.$$

Therefore,

$$\sum_{n=1}^{\infty} \cos(nx)e^{-nt} = \frac{e^{-t}(\cos x - e^{-t})}{1 - 2e^{-t}\cos x + e^{-2t}}$$

and

$$\sum_{n=1}^{\infty} \sin(nx)\, e^{-nt} = \frac{e^{-t}\sin x}{1 - 2e^{-t}\cos x + e^{-2t}}.$$

■

Identity 4.8 For $0 < x < 2\pi$,

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n} = -\ln\left(2\sin\frac{1}{2}x\right).$$

Proof. First, start with the series

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{v}}$$

where $v \in N$ and $0 < x < 2\pi$, if $v = 1$ or $0 \le x \le 2\pi$, if $v > 1$. Using Theorem 3.3

$$\frac{1}{n^{v}} = \frac{1}{(v-1)!}\int_0^{\infty} e^{-nt}t^{v-1}dt,$$

write

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{v}} = \sum_{n=1}^{\infty}\left[\frac{1}{(v-1)!}\int_0^{\infty} \cos(nx)e^{-nt}t^{v-1}dt\right]$$

$$= \frac{1}{(v-1)!}\sum_{n=1}^{\infty}\int_0^{\infty}\cos(nx)e^{-nt}t^{v-1}dt$$

$$= \frac{1}{(v-1)!}\int_0^{\infty}\left(\sum_{n=1}^{\infty}\cos(nx)e^{-nt}\right)t^{v-1}dt$$

$$= \frac{1}{(v-1)!}\int_0^{\infty}\frac{e^{-t}(\cos x - e^{-t})}{1-2e^{-t}\cos x + e^{-2t}}t^{v-1}dt.$$

Suppose that $u = e^{-t}$. Then $du = -e^{-t}dt$. Also, $u = e^{-t}$ implies that $\ln u = -t$. Moreover, $-\ln u = t$. As $t \to \infty$, $u \to 0$ and at $t = 0, u = 1$.

$$\frac{1}{(v-1)!}\int_0^{\infty}\frac{e^{-t}(\cos x - e^{-t})}{1-2e^{-t}\cos x + e^{-2t}}t^{v-1}dt$$

$$= \frac{1}{(v-1)!}\int_1^{0}\frac{\cos x - u}{1-2u\cos x + u^2}(-\ln u)^{v-1}(-du)$$

$$= \frac{(-1)^{v-1}}{(v-1)!}\int_0^{1}\frac{\cos x - u}{1-2u\cos x + u^2}(\ln u)^{v-1}du.$$

To obtain $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}$, substitute $v = 1$. Hence,

$$\frac{(-1)^{v-1}}{(v-1)!}\int_0^{1}\frac{\cos x - u}{1-2u\cos x + u^2}(\ln u)^{v-1}du = \int_0^{1}\frac{\cos x - u}{1-2u\cos x + u^2}du.$$

Let $m = 1 - 2u\cos x + u^2$. Hence, $dm = (-2\cos x + 2u)du$ which further implies that $\frac{-dm}{2} = (\cos x - u)du$. At $u = 1, m = 2 - 2\cos x$ and at $u = 0, m = 1$. Accordingly,

$$\int_0^{1}\frac{\cos x - u}{1-2u\cos x + u^2}du = -\frac{1}{2}\int_1^{2-2\cos x}\frac{dm}{m}$$

$$= -\frac{1}{2}\ln m|_1^{2-2\cos x}$$

$$= -\frac{1}{2}[\ln(2-2\cos x) - \ln 1]$$
$$= -\frac{1}{2}\ln(2-2\cos x)$$
$$= -\ln(2-2\cos x)^{\frac{1}{2}}$$
$$= -\ln\sqrt{\frac{4(1-\cos x)}{2}}$$
$$= -\ln\left(2\sin\frac{1}{2}x\right).$$

Therefore,
$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n} = -\ln\left(2\sin\frac{1}{2}x\right).$$

■

Identity 4.9 For $0 < x < 2\pi$,
$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n} = \arctan\left(\frac{\sin x}{1+\cos x}\right) + \arctan(\cot x).$$
Proof. Note that by Theorem 3.3,
$$\frac{1}{n^v} = \frac{1}{(v-1)!}\int_0^{\infty} e^{-nt}t^{v-1}dt.$$
Now, starting at
$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^v}.$$
Thus,
$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^v} = \sum_{n=1}^{\infty}\sin(nx)\left(\frac{1}{(v-1)!}\int_0^{\infty} e^{-nt}t^{v-1}dt\right)$$
$$= \frac{1}{(v-1)!}\sum_{n=1}^{\infty}\int_0^{\infty}\sin(nx)e^{-nt}t^{v-1}dt$$

$$= \frac{1}{(v-1)!}\int_0^{\infty}\left(\sum_{n=1}^{\infty}\sin(nx)\,e^{-nt}\right)t^{v-1}dt$$
$$= \frac{1}{(v-1)!}\int_0^{\infty}\left(\frac{e^{-t}\sin x}{1-2e^{-t}\cos x+e^{-2t}}\right)t^{v-1}dt.$$

Let $u=e^{-t}, -du=e^{-t}dt$ which implies that $t=-\ln u$. At $t=0, u=1$ and as $t\to\infty$,
$u\to\infty$. Now

$$\frac{1}{(v-1)!}\int_0^{\infty}\left(\frac{e^{-t}\sin x}{1-2e^{-t}\cos x+e^{-2t}}\right)t^{v-1}dt$$
$$= \frac{1}{(v-1)!}\int_0^{1}\frac{\sin x(-\ln u)^{v-1}}{1-2u\cos x+u^2}(-du)$$

$$= \frac{(-1)^{v-1}}{(v-1)!}\int_0^{1}\frac{(\ln u)^{v-1}}{1-2u\cos x+u^2}du.$$

By manipulation, note that $1-2u\cos x+u^2=(u-\cos x)^2+\sin^2 x$.
Hence,

$$\sin x\int_1^0\frac{1}{1-2u\cos x+u^2}du=\sin x\int_1^0\frac{1}{(u-\cos x)^2+(\sin x)^2}du.$$

But

$$\sin x\int_1^0\frac{1}{(u-\cos x)^2+(\sin x)^2}du=\sin x\left[\frac{1}{\sin x}\arctan\left(\frac{u-\cos x}{\sin x}\right)\right]_0^1$$
$$=\arctan\left(\frac{u-\cos x}{\sin x}\right)_0^1$$
$$=\arctan\left(\frac{1-\cos x}{\sin x}\right)-\arctan\left(\frac{-\cos x}{\sin x}\right)$$
$$=\arctan\left(\frac{\sin x}{1+\cos x}\right)+\arctan(\cot x).$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}=\arctan\left(\frac{\sin x}{1+\cos x}\right)+\arctan(\cot x).$$

■

Applications

1. $\sum_{n=1}^{\infty}\frac{1}{n\left(n+\frac{1}{2}\right)}=4(1-\ln 2)$;

2. $\sum_{n=1}^{\infty}\frac{r^n}{n}=\ln\left(\frac{1}{1-r}\right)$;

3. $\sum_{n=1}^{\infty}\frac{r^n}{n(n+1)}=1+\left(\frac{1-r}{r}\right)\ln(1-r)$;

4. For $b\in\{1,2,3\cdots\}$, $\sum_{n=1}^{\infty}\frac{1}{n(n+b)}=\frac{1}{b}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{b}\right)$;

5. $\sum_{n=1}^{\infty}\frac{1}{n(n+5)}=\frac{137}{300}$;

6. $\sum_{n=1}^{\infty}\frac{1}{n4^n}=\ln\left(\frac{4}{3}\right)$;

7. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n+1}=1-\tan^{-1}(1)$;

8. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}=\ln 4-1$;

9. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+2)}=\frac{-1}{4}$;

10. For $0 < x < 2\pi$, $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n} = -\ln\left(2\sin\frac{1}{2}x\right)$.

Conclusion

In this paper, the Laplace transform technique can be applied to infinite series of the forms:

1. $\sum_{n=1}^{\infty} \frac{1}{n+a}$ where $a > -1$;

2. $\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)}$ where $b > a > -1$;

3. $\sum_{n=1}^{\infty} \frac{r^n}{(n+a)(n+b)}$ where $a > 0, b \geq 0, r \in [-1,1)$;
4. $\sum_{n=1}^{\infty} \frac{1}{an+b}$ where $b > a > -1$;

5. $\sum_{n=1}^{\infty} \frac{r^n}{an+b}$ where $a > 0, b \geq 0, r \in [-1,1]$;

6. $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n}$ where $0 < x < 2\pi$;

7. $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ where $0 < x < 2\pi$.

Moreover, proper justifications in the process of switching the sum and the integral are necessary in the technique. Also, for every form of infinite series considered, restrictions to its variables are given in order to use the technique.

As a conclusion, it is not suggested that the closed form expressions for the series discussed in this paper are new. There are other various ways to evaluate them. Rather, it has been the intention of this study to give exposure to a nice technique for evaluating certain forms of series. It is hoped that the readers will add this technique to their toolbox of tricks for infinite series.

Recommendations

The following are recommended for further investigations:

1. For $-\pi < x < \pi$, evaluate the trigonometric infinite series
$$\sum_{n=1}^{\infty}(-1)^n \frac{\cos(nx)}{n}$$
using the Laplace transform technique.

2. For $-\pi < x < \pi$, evaluate the trigonometric infinite series
$$\sum_{n=1}^{\infty}(-1)^n \frac{\sin(nx)}{n}$$
using the Laplace transform technique.

References

[1] Buck, Creighton R. *Advanced Calculus*, Third Edition, pp. 279

[2] Efthimiou, Costas J. *Finding Exact Values for Infinite Series*, Mathematics Magazine, vol. 72, no. 1(Feb., 1999), pp. 45-51.

[3] Efthimiou, Costas J. *Trigonometric Series via Laplace Transform*, Mathematics Magazine, vol. 79, no. 5(Dec., 2009), pp. 376-379.

[4] Gaskill, Herbert S., and Narayanaswani, P.P. *Elements of Real Analysis*, International Edition, pages 336-337.

[5] Leithold, Louis. *The Calculus with Analytic Geometry*, 6th. Edition.

[6] Lesko, James P. and Smith, Wendy S. *A Laplace Transform Technique for Evaluating Infinite Series*, Mathematics Magazine, vol. 76, no. 5(Dec., 2003), pp. 394-398.

[7] Neilsen, Ole A. *An Introduction to Integration and Measure Theory*, pages 111-113.

[8] Rainville, Earl D. and Bedient, Phillip E. *Elementary Differential Equations*, 6th. Edition.

[9] Rudin, Walter. *Principle of Mathematical Analysis*, pages 207, 209-210.

[10] http://www.cimt.plymouth.ac.uk/journal/bagni.pdf

[11] http://en.wikipedia.org/wiki/Laplace-transform

[12] http://www.math.wpi.edu/IQP/BVCalcHist/Calc3.html

[13] http://homepage.newschool.edu/foleyd/GECO6289/laplace.pdf

[14] http://www.efunda.com/webM/transform/laplace.cfm

[15] http://www.efunda.com/webM/transform-inv/laplace.cfm

[16] http://en.wikipedia.org/wiki/Monotone-convergence-theoremcite-note-1