A Note on Convex Basic Graphs

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Abstract

A graph G is convex basic if the convex subsets of the vertex set $V(G)$ of G are all trivial. In this paper, shall give some characterizations of convex basic graphs. Specifically, we shall relate convex basic graphs with the concepts such as hull set and convexity number of a graph. Convex basic graphs resulting from the sum, composition, and cartesian product of graphs are also characterized. As one of our results, we show that for any positive integer p, the set of all connected graphs with independence number p contains only a finite number of convex basic graphs and an infinite number of non-convex basic graphs.

Keywords: graph, convex basic, convex hull, hull number, convexity number

1 Introduction

Given a connected graph G, the *distance* $d_G(x, y)$ is defined as the length of a shortest path connecting vertices x and y of G. It is known that d_G is a metric

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on the vertex set $V(G)$ of G. Any x-y path of length $d_G(x, y)$ is called an x-y *geodesic*. A subset C of $V(G)$ is *convex* if for every two vertices $x, y \in C$, the vertex set of every x-y geodesic is contained in C. The cardinality of a maximum convex proper subset of $V(G)$ is called the convexity number of G and is denoted by $con(G)$. For two vertices x and y of G, the closed interval $I[x, y]$, consists of x and y and all vertices lying on some x-y geodesic in G. If $S \subseteq V(G)$, then $I_G(S) = \cup_{x,y \in S} I(x, y)$. Clearly, the set S is convex in G if $I_G(S) = S$. The *convex hull* of a subset S of $V(G)$, denoted by $[S]$, is the smallest convex set in G containing S. It can be formed from the sequence $\{I^p_G(S)\}$, where p is a non-negative integer, $I^0_G(S) = S$, $I^1_G(S) = I_G(S)$, and $I^p_G(S) = I_G(I^{p-1}_G(S))$ for $p \geq 2$. For some p, we must have $I^q_G(S) = I^p_G(S)$ for all $q \geq p$. If p is the smallest nonnegative integer such that $I^q_G(S) = I^p_G(S)$ for all $q \geq p$, then $I^p_G(S) = [S]$. A subset S of $V(G)$ is a hull set in G if $[S] = V(G)$. The *hull number* $h(G)$ of G is the cardinality of a minimum hull set (a hull set of minimum cardinality) of G. These concepts were introduced by Everett and Seidmann [5] and investigated further in [3], and [4].

2 Definitions and Results

Let G be a connected graph of order $|V(G)| = n \geq 1$. A convex subset S of $V(G)$ is *trivial* if it is one of the following sets:

$$\varnothing,\ \{v\}\ (v \in V(G)),\ \{u, v\}\ (uv \in E(G)) \text{ and } V(G))$$

For convenience, we denote by $TC(G)$ the set containing all the trivial convex subsets of G.

Definition 2.1 A connected graph G of order $n \geq 1$ is *convex basic* if every convex set in G is trivial.

The concept of convex basic graph is defined in [1]. Previous studies on convex basic graphs had been focused mainly on planar graphs (i.e., graphs that can be drawn in the plane with no crossing edges). Planar convex basic graphs were characterized by Hebbare and Rao [7].

Example 2.2 Each of the graphs in Figure 1 is convex basic. The graphs in Figure 2 are not convex basic.

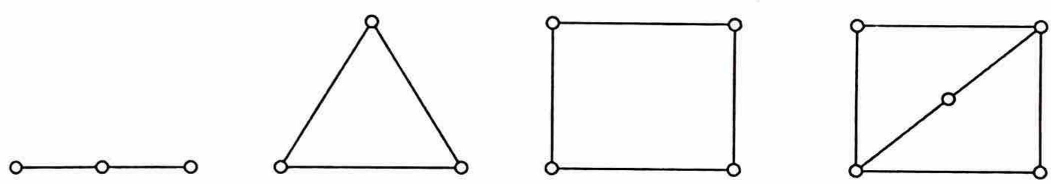

Figure 1 (Convex basic graphs)

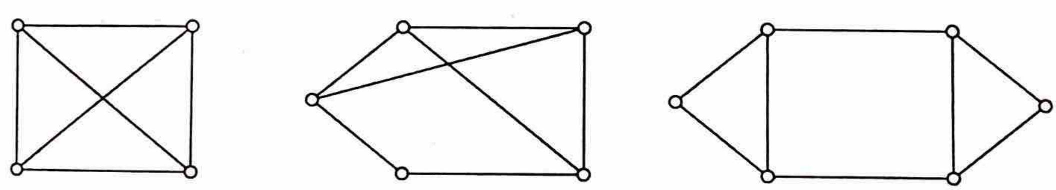

Figure 2 (Non-convex basic graphs)

Our first result uses the concept of hull set to characterize convex basic graphs. In what follows, $P(V(G))$ denotes the family of all subsets of $V(G)$.

Theorem 2.3 *Let G be a connected graph of order $n \geq 1$. Then G is convex basic if and only if every set $S \in P(V(G)) \setminus TC(G)$ is a hull set in G.*

Proof: Suppose G is convex basic and let $S \in P(V(G)) \backslash TC(G)$. Then S is not a convex set in G; hence $I_G(S) \neq S$. Also, for any positive integer p with $I^p_G(S) \neq V(G)$, $I^p_G(S)$ is not convex because G is convex basic. This implies that $[S] = V(G)$, i.e., S is a hull set in G.

Conversely, assume that $[S] = V(G)$ for every $S \in P(V(G)) \backslash TC(G)$. Then $I_G(S) \neq S$ for all $S \in P(V(G)) \backslash TC(G)$. This implies that none of the sets S is convex in G. Therefore G is convex basic. □

The following result is a quick consequence of Theorem 2.3.

Corollary 2.4 *Let G be a connected graph of order $n \geq 3$. If G is convex basic, then*

$$h(G) = \begin{cases} 2, & \text{if } G = P_3 \text{ or } n > 3 \\ 3, & \text{if } G = K_3. \end{cases}$$

Proof: Suppose first that $n > 3$. Since the complete graph K_n $(n \geq 4)$ is not convex basic, $G \neq K_n$. Pick $u, v \in V(G)$ with $uv \notin E(G)$ and let $S = \{u, v\}$. Then $S \notin TC(G)$; hence S is a hull set in G by Theorem 2.1. Since singleton sets are convex sets, none of the singleton subsets of $V(G)$ is a hull set in G. Therefore S is a minimum hull set in G. This proves that $h(G) = 2$.

Next, let $n = 3$. Since P_3 and K_3 are the only two connected graphs of order 3 up to isomorphism, $G = P_3$ or $G = K_3$. If $G = P_3$, then $h(G) = 2$. If $G = K_3$, then $h(G) = 3$. This completes the proof of the corollary. □

The following remark is immediate from the proof of Corollary 2.4.

Remark 2.5 *Let G be a connected convex basic graph of order $n \geq 3$. Then $h(G) = 3$ if and only if $G = K_3$.*

Remark 2.6 *The converse of Corollary 2.4 is not valid.*

To see this, consider the graph G in Figure 3. It is easy to see that the set $S = \{x, y\}$ is the unique minimum hull set in G. Thus $h(G) = 2$. However, this graph is not convex basic because the convex set $C = \{a, b, c, d\}$ is not trivial.

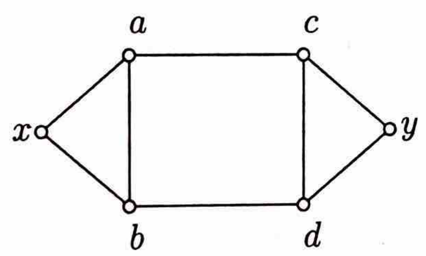

Figure 3

Our next characterization of convex basic graphs involves the concept of convexity number of a graph. This result also characterizes all graphs of convexity number equal to two.

Theorem 2.7 *Let G be a connected graph of order $n \geq 3$. Then G is convex basic if and only if $con(G) = 2$.*

Proof: Suppose G is convex basic. If $n = 3$, then $G = K_3$ or $G = P_3$. Hence, $con(G) = 2$. If $n > 3$, then any set $S \subset V(G)$ with $3 \leq |S| < n$ is not convex in G. Pick $u, v \in V(G)$ with $uv \in E(G)$. Then $C = \{u, v\}$ is convex in G. Accordingly, $con(G) = |C| = 2$.

For the converse, assume that $con(G) = 2$. If $n = 3$, then $G = K_3$ or $G = P_3$; hence G is convex basic. So, suppose $n > 3$. Since $con(G) = 2$,

G has no convex proper subsets S with $|S| > 2$. Clearly, every doubleton subset $\{u, v\}$ of $V(G)$ with $uv \notin E(G)$ is not convex in G. Therefore G is a convex basic graph. □

Using Ramsey numbers, Chartrand and Zhang in [4] obtained the following important result.

Theorem 2.8 *For every graph G of order $n \geq 3$, $con(G) + con(\overline{G}) = 4$ if and only if $n = 3$.*

From this theorem and Theorem 2.7, the following result is immediate.

Corollary 2.9 *If G is a graph of order $n \geq 4$ such that both G and $\overline{G}$ are connected, then at least one of them is non-convex basic.*

The *girth* $g(G)$ of a graph G is the length of a shortest cycle, if any exists, in G. Clearly, any graph G containing K_3 as a proper subgraph is not convex basic. From this observation, we obtain the following remark.

Remark 2.10 *Every connected graph G with $g(G) = 3$ that is not isomorphic to K_3 is non-convex basic.*

Suppose now that G is a connected graph with $g(G) = t > 4$. Let $C_t = [x_1, x_2, ..., x_t, x_1]$ be a cycle in G of length t. Consider the set $S = \{x_1, x_2, x_3\}$. Since $t > 4$, $d_G(x_1, x_3) = 2$. If there exists an $x \in V(G)\backslash S$ such that $x \in I[x_1, x_3]$, that is, x is a vertex in an x_1-x_3 geodesic, then the set $\{x_1, x_2, x_3, x\}$ induces a cycle of length 4. This implies that $g(G) \leq 4$, contrary to our assumption. Thus we have proved the following remark.

Remark 2.11 *If G is a connected graph with $g(G) > 4$, then G is non-convex basic.*

We shall now consider the sum, the composition and the Cartesian product of graphs. It is worth mentioning that it may happen that none of the graphs $G+H$, $G[H]$, and $G \times H$ is convex basic even if the graphs G and H are convex basic graphs. As an example, consider the convex basic graphs $G = C_4$ and $H = K_2$. It can be verified that $con(G+H) = 5$, $con(G[H]) = 4$, and $con(G \times H) = 4$ (see [2]). Thus none of the graphs $G+H$, $G[H]$, and $G \times H$ is convex basic by Theorem 2.7.

We can get more examples using the following result.

Lemma 2.12 *Let G and H be non-trivial graphs. If G or H has at least one non-trivial component, then $G+H$ is non-convex basic.*

Proof: Without loss of generality, assume that G has a non-trivial component, say G^*. Choose $u, v \in V(G^*)$ and $x \in V(H)$ such that $uv \in E(G)$. Then the set $S = \{u, v, x\}$ induces a complete proper subgraph of G by definition of $G^* + H$. It follows that $g(G+H) = 3$. Therefore $G+H$ is non-convex basic by Remark 2.10. □

Theorem 2.13 *Let G and H be graphs. Then $G+H$ is convex basic if and only if either*

(a) $2 \leq |V(G+H)| \leq 3$, *or*

(b) *G and H are non-trivial and all their components are trivial.*

Proof: Assume that $G + H$ is convex basic. Suppose that (a) does not hold. Then $|V(G+H)| \geq 4$. If G or H is trivial, say G, then H is non-trivial. Consider the following cases:

Case 1. Suppose H is connected. Since $|V(G + H)| \geq 4$, $|V(H)| \geq 3$. Let $uv \in E(H)$ and H^* be the subgraph of H induced by $S = \{u, v\}$. Then $G + H^* \cong K_3$ is a complete proper subgraph of $G + H$. It follows that $C = V(G) \cup S$ is a non-trivial convex set in $G + H$. This contradicts our assumption that $G + H$ is convex basic.

Case 2. Suppose H is disconnected with components $H_1, H_2, ...H_r$. If a component of H, say H_k, is non-trivial, then $V(G + H_k)$ is a non-trivial convex subset of $V(G + H)$. This is not possible because $G + H$ is convex basic. Therefore all the components of H are trivial. Further, since $G + H$ is not (isomorphic to) the path P_3, $r \geq 3$. Let H^* be the disjoint union of H_1 and H_2. Then $G + H^* \cong K_3$; hence $V(G + H^*)$ is a non-trivial convex set $G + H$. Again, this gives a contradiction to our assumption.

Therefore G and H are both non-trivial graphs. Moreover, the components of G and H are all trivial by the contrapositive of Lemma 2.1. This shows that condition (b) holds.

For the converse, suppose first that (a) holds. Then either $G + H \cong K_2$, or $G + H \cong K_3$, or $G + H \cong P_3$. In any of these cases we find that $G + H$ is convex basic. Next, suppose that (b) holds. Let $C = S_1 \cup S_2$ be a convex set in $G + H$, where $card(C) \geq 3$, $S_1 \subseteq V(G)$, and $S_2 \subseteq V(H)$. Then either $card(S_1) \geq 2$ or $card(S_2) \geq 2$. Without loss of generality, assume that $card(S_1) \geq 2$. Let $x, y \in S_1$. Then $d_{G+H}(x, y) = 2$. This implies that $h \in I[x, y]$ for every $h \in V(H)$. By convexity of C, it follows that $V(H) \subset C$;

hence $S_2 = V(H)$. Since $|V(H)| \geq 2$, we can use a similar argument to show that $S_1 = V(G)$. Therefore

$$C = V(G) \cup V(H) = V(G+H) .$$

Accordingly, $G+H$ is a convex basic graph. □

The following are consequences of Theorem 2.13.

Corollary 2.14 *Let G and H be connected graphs. Then $G+H$ is convex basic if and only if either $G+H \cong K_2$ or $G+H \cong K_3$.*

Corollary 2.15 *Every complete bipartite graph $K_{m,n}$ is convex basic except $K_{1,n}$, where $n \geq 3$.*

Proof: Clearly, $K_{m,n} = \overline{K_m} + \overline{K_n}$. Thus, by Theorem 2.13, $K_{m,n}$ is convex basic for any positive integers m and n except when $m = 1$ and $n \geq 3$ (or $m \geq 3$ and $n = 1$). This proves the corollary. □

Corollary 2.16 *Let G be a connected graph of order $n \geq 4$ which is neither a complete graph nor a complete bipartite graph. If $\overline{G}$ is disconnected, then G is non-convex basic.*

Proof: Since G is non-complete, then not all components of $\overline{G}$ are trivial. Let $H_1, H_2, \ldots, H_t$ be the components of $\overline{G}$. Clearly,

$$G = \overline{H_1} + \overline{H_2} + \ldots + \overline{H_t} .$$

If $t \geq 3$, then K_3 is a subgraph of G. Since K_3 is convex, G is non-convex basic. If $t = 2$, then $G = \overline{H_1} + \overline{H_2}$. Since G is not a complete bipartite graph, not both H_1 and H_2 can be complete graphs. Assume that H_1 is not complete. Then not all the components of $\overline{H_1}$ are trivial. Therefore G is non-convex basic by Theorem 2.8. □

Corollary 2.17 *Let G be a connected graph of order $n \geq 6$ and size m. If $\binom{n-1}{2} \leq m < \binom{n}{2}$, then G is non-convex basic.*

Proof: Since $m < \binom{n}{2}$, G is not a complete graph. Suppose that G is a complete bipartite graph, say $G = K_{p,q}$, where $p + q = n$. Then $m = pq$ and so

$$\binom{n-1}{2} \leq pq < \binom{n}{2} .$$

This implies that

$$p(n-p) \geq \frac{(n-1)(n-2)}{2} .$$

This gives the quadratic inequality $2p^2 - 2np + (n-1)(n-2) \leq 0$. Now, the graph of

$$f(p) = 2p^2 - 2np + (n-1)(n-2)$$

is a parabola which is concave upward. The vertex occurs at $p = \frac{n}{2}$ and the value of f at this value of p is

$$\frac{n^2}{2} - 3n + 2 ,$$

which is positive for all $n \geq 6$. Thus, the inequality $f(p) \leq 0$ cannot hold. Therefore G is not a complete bipartite graph.

Finally, we shall show that $\overline{G}$ is disconnected. To this end, suppose $\overline{G}$ is connected. Then it has a spanning tree with $n-1$ edges. It follows that the size of G does not exceed

$$\binom{n}{2} - (n-1) < \frac{(n-1)(n-2)}{2} .$$

This is a contradiction. Therefore $\overline{G}$ is disconnected. By Corollary 2.16, G is non-convex basic. □

The clique number $\omega(G)$ of a graph G is the cardinality of a largest clique in G. We use this concept in our next result.

Theorem 2.18 *Let G and H be connected graphs. If $\omega(G) \geq 2$ and $\omega(H) \geq 2$, then the composition $G[H]$ of G and H is non-convex basic.*

Proof: Suppose $\omega(G) \geq 2$ and $\omega(H) \geq 2$. If $G[H]$ is (isomorphic to) K_4, then it is non-convex and we are done. So suppose that $G[H]$ is not K_4. Choose $u, v \in V(G)$ and $x, y \in V(H)$ such that $uv \in E(G)$ and $xy \in E(H)$. Let $S = \{(u,x),(u,y),(v,x),(v,y)\}$. Then S induces a subgraph isomorphic to K_4, by definition of $G[H]$. Hence, S is a convex set in $G[H]$. This means that $V(G[H])$ contains a non-trivial convex set S. Accordingly, $G[H]$ is non-convex basic. □

Theorem 2.19 *Let G and H be connected graphs. Then $G[H]$ is convex basic if and only if either $G = K_1$ and H is convex basic or $H = K_1$ and G is convex basic.*

Proof: Suppose $G[H]$ is convex basic. Then either $G = K_1$ or $H = K_1$ by the contrapositive of Theorem 2.18. If $G = K_1$, then $G[H]$ is isomorphic to H. Thus, H is convex basic by assumption. Similarly, G is convex basic if $H = K_1$. Therefore, either $G = K_1$ and H is convex basic or $H = K_1$ and G is convex basic.

Conversely, suppose that either $G = K_1$ and H is convex basic or $H = K_1$ and G is convex basic. Then $G[H]$ is either (isomorphic to) H or G. In each of these cases, $G[H]$ is convex basic.

This completes the proof of the theorem. □

We shall use the following result in [2].

Theorem 2.20 *Let G and H be connected graphs. Then*

$$con(G \times H) = max\{|V(G)|con(H), |V(H)|con(G)\}.$$

Theorem 2.21 *Let G and H be connected graphs. Then $G \times H$ is convex basic if and only if either $G = K_1$ and H is convex basic or $H = K_1$ and G is convex basic.*

Proof: If $|V(G \times H)| \leq 2$, then $G \times H$ is convex basic and either $G \times H = K_1$ or $G \times H = K_2$. Hence either $G = K_1$ and $H = K_1$ (convex basic) or $H = K_1$ and $G = K_2$ (convex basic) or $G = K_1$ and $H = K_2$ (convex basic).

Suppose $|V(G \times H)| \geq 3$. Then $G \times H$ is convex basic if and only if $con(G \times H) = 2$ by Theorem 2.7. Now, $con(G \times H) = 2$ implies that either $|V(G)|con(H) = 2$ or $|V(H)|con(G) = 2$ and none of these two has value more than two, by Theorem 2.20. If $|G|con(H) = 2$, then either $|V(G)| = 1$ and $con(H) = 2$ or $|V(G)| = 2$ and $con(H) = 1$. Clearly, $|V(G)| = 1$ and $con(H) = 2$ if and only if $G = K_1$ and H is convex basic by Theorem 2.7. Now, $|V(G)| = 2$ and $con(H) = 1$ if and only if $G = K_2$ and $H = K_1$. Thus, $|V(G)|con(H) = 2$ if and only if either $G = K_1$ and H is convex basic or $G = K_2$ and $H = K_1$. Similarly, $|V(H)|con(G) = 2$ if and only if either

$H = K_1$ and G is convex basic or $H = K_2$ and $G = K_1$. Therefore, since K_1 and K_2 are convex basic, it follows that $G \times H$ is convex basic if and only if either $G = K_1$ and H is convex basic or $H = K_1$ and G is convex basic. □

For any connected graph G, we denote by $\alpha(G)$ the (vertex) independence number of G and by $\text{Ind}(p)$ the set of all connected graphs with independence number p. We shall show that $\text{Ind}(p)$ contains only a finite number of convex basic graphs for every positive integer p. Our result is actually motivated by the following theorem:

Theorem 2.22 *Let G be a connected graph of order $n \geq 6$. If $\alpha(G) = 2$, then G is not convex basic.*

Proof: Let $S = \{x, y\}$ be an independent set. Observe that since $\alpha(G) = 2$, $\text{dist}(x, y) = 2$ or $\text{dist}(x, y) = 3$. Consider the following cases:

Case 1. Suppose $\text{dist}(x, y) = 2$. Let $T = \{a_1, a_2, ..., a_t\}$ be the set consisting all the common neighbors of x and y. If $t \geq 3$, then T cannot be independent. Without loss of generality, assume that $a_1a_2 \in E(G)$. Then $\{a_1, a_2, x\}$ forms a complete graph in G. Thus, G is non-convex basic. If $t = 1$ then $\{x, y, a_1\}$ is convex in G; hence, G is non-convex basic. If $t = 2$, then there exists $b \notin T \cup S$. Since $\alpha(G) = 2$, either $xb \in E(G)$ or $yb \in E(G)$ but not both. Assume that $xb \in E(G)$. If $a_1a_2 \in E(G)$, then $\{a_1, a_2, x\}$ forms a complete graph in G. If $a_1a_2 \notin E(G)$, then either $a_1b \in E(G)$ or $a_2b \in E(G)$, say $a_1b \in E(G)$. Then $\{a_1, b, x\}$ forms a complete graph in G. This shows that G is not convex basic.

Case 2. Suppose $\text{dist}(x, y) = 3$. Let $P = [x, a_1, a_2, y]$ be an x-y geodesic.

Let $b \in V(G)\backslash V(P)$. Since $\{x, y, b\}$ cannot be independent, b is adjacent to either x or y but not both. Assume that $xb \in E(G)$. Since $\{a_1, b, y\}$ cannot be independent, $ba_1 \in E(G)$. Thus $\{x, a_1, b\}$ induces the complete graph K_3. Therefore G is non-convex basic. □

Lemma 2.23 *$Ind(p)$ is an infinite set for every positive integer p.*

Proof: $Ind(1)$ is the set of all complete graphs which is infinite. Let $p \geq 2$. For every $q \geq 1$, the graph $K_q + \overline{K_p}$ is clearly connected and has independence number p. Thus $Ind(p)$ is an infinite set. □

We shall need the following result from Ramsey Theory to establish our last result.

Theorem 2.24 *For each integer $p \geq 1$, there exists a number $r(p))$ such that every graph of order $n \geq r(p)$ either has a complete subgraph K_p or an independent subset S with $|S| = p$.*

Theorem 2.25 *For each integer $p \geq 1$, $Ind(p)$ contains only a finite number of convex basic graphs and an infinite number of non-convex basic graphs.*

Proof: $Ind(1)$ is the set of complete graphs. Only K_1, K_2 and K_3 are convex basic among the complete graphs. Let $p \geq 2$ and suppose that Ind(p) contains infinitely many convex basic graphs. Then there exist convex basic graphs in Ind(p) with arbitrarily large orders. Let $N = p + r(p+1)$, where $r(p+1)$ is the Ramsey number corresponding to $p+1$ in Theorem 2.24. Then there exists a graph $G \in$ Ind(p) of order $n \geq N$. Let S be an independent set in G with $|S| = p$ and consider the graph H induced by $V(G)\backslash S$. Since the order of H is

$$n - p \geq N - p = r(p+1) \,,$$

either H contains a subgraph K_{p+1} or an independent subset with $p+1$ elements by Theorem 2.22. Since $\alpha(G) = p$, H cannot have an independent subset of order $p+1$. Thus, H has a subgraph K_{p+1}. But this subgraph is a non-trivial convex subset of G. This contradicts our assumption that G is convex basic. Therefore, $Ind(p)$ contains only a finite number of convex basic graphs. By Lemma 2.23, it follows that Ind(p) contains infinitely many non-convex basic graphs. □

Finally, we pose the following conjecture for the interested readers.

Conjecture. Let $\gamma(n)$ denote the number of convex basic graphs of order n and let $\Gamma(n)$ denote the total number of non-isomorphic connected graphs of order n. Then

$$\lim_{n\to\infty} \frac{\gamma(n)}{\Gamma(n)} = 0.$$

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