The Orthogonality and Inverse Relations of the Limit of Differences of the Generalized Factorial

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Abstract

The orthogonality and inverse relations of $s(n,k)$ and $S(n,k)$ indicate that the numbers $F_{\alpha,\gamma}(n,k)$ and $\left\langle {n \atop k} \right\rangle_{\beta,\gamma}$ may satisfy certain orthogonality and inverse relations. With the aid of the horizontal generating function for $F_{\alpha,\gamma}(n,k)$ and the definition of $\left\langle {n \atop k} \right\rangle_{\beta,\gamma}$, the orthogonality and inverse relations of $F_{\alpha,\gamma}(n,k)$ and $\left\langle {n \atop k} \right\rangle_{\beta,\gamma}$ will be established.

Keywords: orthogonality, generating function, limit, generalized factorial

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1 Introduction

The limit of the differences of the generalized factorial, denoted by $F_{\alpha,\gamma}(n,k)$,

$$F_{\alpha,\gamma}(n,k) = \lim_{\beta\to 0} \frac{\left[\Delta_t^k(\beta t+\gamma|\alpha)_n\right]_{t=0}}{k!\beta^k}$$

was evaluated completely in [4]. The limit gives the explicit formula

$$F_{\alpha,\gamma}(n,k) = \sum_{0\le j_1<j_2<\cdots<j_{n-k}\le n-1} \prod_{q=1}^{n-k}(\gamma - j_q\alpha) \ ,$$

and, consequently, obtains the following properties; the triangular recurrence relation

$$F_{\alpha,\gamma}(n+1,k) = F_{\alpha,\gamma}(n,k-1) + (\gamma - n\alpha)F_{\alpha,\gamma}(n,k) \ ,$$

the horizontal generating function

$$\sum_{k=0}^{n} F_{\alpha,\gamma}(n,k)t^k = (t+\gamma|\alpha)_n \ ,$$

and the vertical generating function

$$(1+\alpha t)^{\gamma/\alpha}\left[\log(1+\alpha t)^{1/\alpha}\right]^k = k!\sum_{n\ge 0} F_{\alpha,\gamma}(n,k)\frac{t^n}{n!} \ .$$

Based on these properties, it can easily be seen that $F_{\alpha,\gamma}(n,k)$ is a generalization of the Stirling number of the first kind $s(n,k)$ (see [1], [2], [5] for further discussion of $s(n,k)$). In particular, $s(n,k) = F_{1,0}(n,k)$. On the other hand, the (r,β)-Stirling number, denoted by $\left\langle {n \atop k} \right\rangle_{\beta,\gamma}$, was defined in [3] by means of the following linear transformation:

$$t^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,\gamma} (t-r)_{\beta,k}$$

where $(t-r)_{\beta,k} = (t-r|\beta)_k$. This number is a certain generalization of Stirling number of the second kind $S(n,k)$. That is, $S(n,k) = \left\langle {n \atop k} \right\rangle_{1,0}$. For a detailed discussion of the (r,β)-Stirling number, one may see [3].

2 Main Results

Note that the horizontal generating function for $F_{\alpha,\gamma}(n,k)$ and the definition of $\left\langle {n \atop k} \right\rangle_{\beta,\gamma}$ can be written as

$$(t-r|\beta)_m = \sum_{k=0}^{n} F_{\beta,-r}(m,k)t^k \tag{1}$$

$$t^k = \sum_{n=0}^{k} \left\langle {k \atop n} \right\rangle_{\beta,r} (t-r|\beta)_n \,, \tag{2}$$

respectively. Hence, substituting (2) in (1) yields

$$\begin{aligned}
(t-r|\beta)_m &= \sum_{k=0}^{m} F_{\beta,-r}(m,k) \left[\sum_{n=0}^{k} \left\langle {k \atop n} \right\rangle_{\beta,r} (t-r|\beta)_n \right] \\
(t-r|\beta)_m &= \sum_{k=0}^{m} F_{\beta,-r}(m,k) \left[\sum_{n=0}^{k} \left\langle {k \atop n} \right\rangle_{\beta,r} (t-r|\beta)_n \right] \\
&= \sum_{k=n}^{m} \left[\sum_{k=n}^{m} F_{\beta,-r}(m,k) \left\langle {k \atop n} \right\rangle_{\beta,r} \right] (t-r|\beta)_n \,.
\end{aligned}$$

Thus,

$$\sum_{k=n}^{m} F_{\beta,-r}(m,k) \left\langle {k \atop n} \right\rangle_{\beta,r} = \delta_{m,n} = \begin{cases} 1, & m=n \\ 0, & m \neq n \end{cases} .$$

This result is embodied in the following theorem.

Theorem 2.1 *The following orthogonality relations hold:*

$$\sum_{k=n}^{m} F_{\beta,-r}(m,k) \left\langle {k \atop n} \right\rangle_{\beta,r} = \sum_{k=n}^{m} \left\langle {m \atop k} \right\rangle_{\beta,r} F_{\beta,-r}(k,n) = \delta_{mn}$$

where δ_{mn} is the Kronecker delta.

Proof: The proof for the first equality is already given above. We are left to show the second equality. Note that the definition of $\left\langle {n \atop k} \right\rangle_{\beta,\gamma}$ can be written as

$$t^m = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle_{\beta,r} (t-r|\beta)_k .$$

With the aid of the horizontal generating function for $F_{\alpha,\gamma}$, we obtain

$$\begin{aligned} t^m &= \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle_{\beta,r} \left[\sum_{n=0}^{k} F_{\beta,-r}(k,n) t^n \right] \\ &= \sum_{n=0}^{m} \left[\sum_{k=n}^{m} \left\langle {m \atop k} \right\rangle_{\beta,r} F_{\beta,-r}(k,n) \right] t^n . \end{aligned}$$

This precisely gives the second equality. □

Remark 2.2 *Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be two* $n \times n$ *matrices whose entries are the numbers* $F_{\alpha,\gamma}(i,j)$ *and* $\left\langle {i \atop j} \right\rangle_{\beta,r}$. *More precisely,*

$$\mathbf{M}_1 = \begin{bmatrix} F_{\beta,-r}(0,0) & 0 & \cdots & 0 \\ F_{\beta,-r}(1,0) & F_{\beta,-r}(1,1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ F_{\beta,-r}(n,0) & F_{\beta,-r}(n,1) & \cdots & F_{\beta,-r}(n,n) \end{bmatrix},$$

$$\mathbf{M}_2 = \begin{bmatrix} \left\langle {0 \atop 0} \right\rangle_{\beta,r} & 0 & \cdots & 0 \\ \left\langle {1 \atop 0} \right\rangle_{\beta,r} & \left\langle {1 \atop 1} \right\rangle_{\beta,r} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \left\langle {n \atop 0} \right\rangle_{\beta,r} & \left\langle {n \atop 1} \right\rangle_{\beta,r} & \cdots & \left\langle {n \atop n} \right\rangle_{\beta,r} \end{bmatrix}$$

Then, using Theorem 2.1, the product of the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ gives

$$\mathbf{M}_1\mathbf{M}_2 = \mathbf{M}_2\mathbf{M}_1 = \mathbf{I}_n ,$$

where I_n is an $n \times n$ identity matrix. This implies that $M_2 = M_1^{-1}$ and $M_1 = M_2^{-1}$, that is, M_1 and M_2 are orthogonal matrices.

The next theorem gives the inverse relation of $F_{\alpha,\gamma}(n,k)$ and $\left\langle {n \atop k} \right\rangle_{\beta,r}$.

Theorem 2.3 *With $n \in \mathbb{N}$ (set of natural numbers), the following inverse relations hold:*

$$f_n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} g_k \Leftarrow: g_n = \sum_{k=0}^{n} F_{\beta,-r}(n,k) f_k \,.$$

Proof: $(=:)$ Using the hypothesis, we have

$$\begin{aligned} \sum_{k=0}^{n} F_{\beta,-r}(n,k) f_k &= \sum_{k=0}^{n} F_{\beta,-r} \left\{ \sum_{j=0}^{k} \left\langle {k \atop j} \right\rangle_{\beta,r} g_j \right\} \\ &= \sum_{j=0}^{n} \left\{ \sum_{k=j}^{n} F_{\beta,-r}(n,k) \left\langle {k \atop j} \right\rangle_{\beta,r} \right\} g_j \,. \end{aligned}$$

By Theorem 2.1, we obtain

$$\sum_{k=0}^{n} F_{\beta,-r}(n,k) f_k = \sum_{j=0}^{n} \delta_{n,j} g_j = g_n \,.$$

$(\Leftarrow)$ Similarly, using the hypothesis,

$$\begin{aligned} \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} g_k &= \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} \left\{ \sum_{j=0}^{k} F_{\beta,-r}(k,j) f_j \right\} \\ &= \sum_{j=0}^{n} \left\{ \sum_{k=j}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} F_{\beta,-r}(k,j) \right\} f_j \\ &= \sum_{j=0}^{n} \delta_{n,j} f_j = f_n \,. \qquad \square \end{aligned}$$

Remark 2.4 *When $\beta = 1$ and $r = 0$, Theorems 2.1 and 2.3 will reduce to the orthogonality and inverse relations of the ordinary Stirling numbers of the first and second kinds, which are given as follows:*

$$\sum_{k=n}^{m} |s(m,k)| S(k,n) = \sum_{k=n}^{m} S(m,k) |s(k,n)| = \delta_{mn} ,$$

$$f_n = \sum_{k=0}^{n} S(n,k) g_k \Leftarrow: g_n = \sum_{k=0}^{n} |s(n,k)| f_k .$$

where

$$|s(n,k)| = (-1)^{n-k} s(n,k)$$

is the signless Stirling number of the first kind.

Remark 2.5 *Note that when $\beta = 0$ and $r = 1$, equation (2) will give*

$$\left\langle {k \atop n} \right\rangle_{0,1} = \binom{k}{n}$$

and, with t being replaced by $t + r$, equation (1) will give

$$F_{0,-1}(n,k) = (-1)^{n-k} \binom{n}{k}$$

where $\binom{n}{k}$ is a binomial coefficient. Then Theorem 2.1 and 2.3 will reduce to the following orthogonality and inverse relations

$$\sum_{k=n}^{m} (-1)^{m-k} \binom{m}{k} \binom{k}{n} = \sum_{k=n}^{m} (-1)^{n-k} \binom{m}{k} \binom{k}{n} = \delta_{mn} ,$$

$$f_n = \sum_{k=0}^{n} \binom{n}{k} g_k \Leftarrow: g_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_k .$$

The preceding inverse relation is the well-known δ *transform*, which is used by G.H. Hardy to introduce Hausdorff summability method. This inverse relation is also useful in simplifying solution of some combinatorial problems. A simple example is the solution of derangement problem (also known as *le problème des rencontres*). The problem is to find the number D_n of permutations $(a_1, a_2, \ldots, a_n)$ of $1, 2, 3, \ldots, n$ such that $a_i \neq i, \forall i = 1, 2, \ldots, n$. It can easily be seen that

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_k .$$

Hence, by making use of the inverse relation (the δ *transform*) with $f_n = n!$ and $g_k = D_k$, we obtain

$$D_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} .$$

which further gives $D_n \approx n!e^{-1}$, for a very large value of n. Note that this problem is solved in [1] using the principle of inclusion and exclusion.

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