On the Geodetic and Hull Numbers of Some Graphs

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Abstract

Let G be a connected graph, u and v be vertices of G and $I[u,v]$ the closed interval consisting of u, v and all vertices lying on some u-v geodesic. If $S \subseteq V(G)$, then $I[S]$ is the union of all sets $I[u,v]$ for all $u, v \in S$. A subset S of $V(G)$ is called a geodetic set in G if $I[S] = V(G)$. The minimum cardinality of a geodetic set in G is called the geodetic number of G. The convex hull $[S]$ of a subset S of $V(G)$ is defined as the smallest convex set in G containing S. The minimum cardinality among the subsets S of $V(G)$ with $[S] = V(G)$ is called the hull number of G. In this paper, we give the geodetic number and the hull number of some graphs.

Keywords: graph, geodetic number, hull number, gluing, deletion

1 Preliminaries

Let G be a connected simple graph and $d_G(x,y)$ be the length of a shortest path connecting two vertices x and y of G. The couple $(V(G), d_G)$, where

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$V(G)$ is the vertex set of G, is a metric space. An x-y path, denoted by $P(x,y)$, of length $d_G(x,y)$ is called an *x-y geodesic*. A subset S of $V(G)$ is *convex* if for every two vertices $x, y \in S$, the vertex set of every x-y geodesic is contained in S. For every two vertices x and y of G, the symbol $I[x,y]$ is used to denote the set consisting of x, y and all vertices lying on some x-y geodesic. A subset S of $V(G)$ is called a *geodetic set* in G if $I[S] = V(G)$, where $I[S] = \cup_{x,y \in S} I[x,y]$. The *geodetic number* of a connected graph G is defined as the cardinality of a minimum geodetic set. It can easily be verified that $S \subseteq I[S]$ and that $I[S] = S$ if and only if S is convex. Convexity in graphs was discussed in [16]. This concept was investigated in [2], [3], [7], [11] and [12]. Some related results on the concept of geodetic number of a graph were obtained in [1], [5], [8], [9], and [15].

The convex hull of a subset S of $V(G)$, denoted by $[S]$, is the smallest convex set in G containing S. It can be formed from the sequence $\{I^p[S]\}$, where p is a non-negative integer, $I^0[S] = S$, $I^1[S] = I[S]$, and $I^p[S] = I[I^{p-1}[S]]$ for $p \geq 2$. For some p, we must have $I^q[S] = I^p[S]$ for all $q \geq p$. Further, if p is the smallest non-negative integer such that $I^q[S] = I^p[S]$ for all $q \geq p$, then $I^p[S] = [S]$.

A subset S of $V(G)$ is a *hull set* in G if $[S] = V(G)$. A hull set of minimum cardinality is called a *minimum hull set* in G. The *hull number* $h(G)$ of G is the cardinality of a minimum hull set in G. These concepts were introduced by Everett and Seidmann [13] and were investigated in [4], [5], [6] and [10].

For other graph theoretic terms which are assumed here, readers are advised to refer to [14].

The following two lemmas were stated in [2].

Lemma 1.1 [2] *Let G be a connected graph. Then $g(G) = |V(G)|$ if and only if G is a complete graph.*

Lemma 1.2 [2] *Let G be a connected graph. Then $h(G) = |V(G)|$ if and only if G is a complete graph.*

Proof: Suppose G is a complete graph. If $S \subseteq V(G)$, then S induces a complete subgraph of G. It follows that S is convex in G and hence, $S = [S]$. This means that $[S] = V(G)$ if and only if $S = V(G)$. Therefore, $h(G) = |V(G)|$.

Conversely, suppose $h(G) = |V(G)|$. Assume further that G is not complete. Then there exist vertices $a, b \in V(G)$ such that $d_G(a, b) = 2$. Let c be a vertex in some a-b geodesic that is distinct from a and b. Set $S = V(G)\backslash\{c\}$. Clearly, S is not convex in G. Also, because c is in some a-b geodesic, it follows that $c \in I[S] \subseteq [S]$. Thus, $[S] = V(G)$.

Therefore $h(G) \leq |S| = |V(G)| - 1$. This contradicts our assumption that $h(G) = |V(G)|$. Therefore G is a complete graph. □

A vertex v in a connected graph G is an *extreme vertex* if the set of neighbors $N(v)$ of v induces a complete subgraph of G. In what follows, $Ex(G)$ denotes the set of extreme vertices in G.

Theorem 1.3 *Let G be a connected graph. If $x \in Ex(G)$, then $V(G)\backslash\{x\}$ is a convex set.*

Proof: Let $x \in Ex(G)$ and let $a, b \in C = V(G)\backslash\{x\}$. If a and b are neighbors of x, then $d_G(a, b) = 1$ and so, the vertex set of every a-b geodesic is contained in C.

Suppose now that none of a and b is a neighbor of x. Clearly, the vertex set of every a-b geodesic is contained in C if $d_G(a,b) \leq 3$. So, suppose $d_G(a,b) \geq 4$. Suppose further that x is in some a-b geodesic $P_{r+2}(a,b) = [a, x_1, \ldots, x_r, b]$ ($r \geq 3$). Then there exists $k \in \{1, 2, \ldots, r-2\}$ such that $x_{k+1} = x$. This implies that x_k and x_{k+2} are neighbors of x. Since $x \in Ex(G)$, $x_k x_{k+2} \in E(G)$. Thus, $[a, x_1, \ldots, x_k, x_{k+2}, \ldots, x_r, b]$ is a path connecting a and b. This contradicts the fact that $P_{r+2}(a,b)$ is an a-b geodesic. Therefore x cannot be a vertex in any a-b geodesic.

If one of a and b is a neighbor of x, then a slightly similar argument as above can be used to prove that the vertex set of every a-b geodesic is contained in C.

Therefore, $V(G)\backslash\{x\}$ is a convex set. □

2 K_r-gluing of Complete Graphs

Definition 2.1 Let $K_{p_1}, K_{p_2}, \ldots$ and K_{p_n} be complete graphs, each containing a complete subgraph K_r ($r \geq 1$). The graph G obtained from the union of these n complete graphs by identifying the K_r's (one from each complete graph) in an arbitrary way is called the K_r-*gluing* of $K_{p_1}, K_{p_2}, \ldots$ and K_{p_n}.

Remark 2.2 *Let p, q and r be positive integers such that $1 \leq r \leq p \leq q$. If G is the K_r-gluing of K_p and K_q, then*

$$Ex(G) = \begin{cases} V(G) & \text{if } r = p, \\ V(G)\backslash V(K_r) & \text{if } r < p. \end{cases}$$

Theorem 2.3 *Let p, q and r be positive integers such that $1 \leq r \leq p \leq q$. If G is the K_r-gluing of K_p and K_q, then*

$$g(G) = \begin{cases} q & \text{if } p = r, \\ q + p - 2r & \text{if } p > r. \end{cases}$$

Proof: Consider the following cases:

Case 1. Suppose $p = r$. Then $G = K_q$. Thus, $g(G) = g(K_q) = q$ by Lemma 1.1.

Case 2. Let $p > r$ and $x \in V(G)$. If $x \in Ex(G)$, then $x \in I[Ex(G)]$. If $x \notin Ex(G)$, then $x \in V(G) \backslash Ex(G) = V(K_r)$. Let $u \in [V(K_q) \backslash V(K_r)]$ and $v \in [V(K_p) \backslash V(K_r)]$. Note that $d_G(u, v) = 2$. Thus, $[u, x, v]$ is a u-v geodesic; that is, $x \in I[u, v]$. Since $u, v \in Ex(G)$, we have $x \in I[Ex(G)]$. Hence, $I[Ex(G)] = V(G)$.

Therefore, by definition, $g(G) = |Ex(G)| = q + p - 2r$. □

Theorem 2.4 *Let p, q and r be positive integers such that $1 \leq r \leq p \leq q$. If G is the K_r-gluing of K_p and K_q, then*

$$h(G) = \begin{cases} q & \text{if } p = r, \text{ and} \\ q + p - 2r & \text{if } p > r. \end{cases}$$

Proof: One can follow the proof of Theorem 2.3. □

The following result is an extension of Theorem 2.3.

Theorem 2.5 *Let $p_1, p_2, \ldots, p_n$ and r be positive integers such that $1 \leq r < p_1 \leq p_2 \leq \cdots \leq p_n$. If G is the K_r-gluing of $K_{p_1}, K_{p_2}, \ldots,$ and K_{p_n}, then $g(G) = \sum_{i=1}^{n} p_i - nr$.*

Proof: As in the proof of Theorem 2.3 (Case 2), we can show that $S_0 = [V(K_{p_1})\backslash V(K_r)] \cup \cdots \cup [V(K_{p_n})\backslash V(K_r)]$ is a minimum geodetic set of G. Therefore, by definition, the result now follows. □

3 Graphs Obtained by Deletion of Edges

Definition 3.1 Let K_n be the complete graph of order $n \geq 3$ and Ω a family of complete proper subgraphs of K_n. We say that Ω is an *independent set* (family) if no two distinct subgraphs in Ω have a common vertex.

Definition 3.2 Let K_n be the complete graph of order $n \geq 3$ and Ω an independent family of complete proper subgraphs of K_n, each of order at least 2. The graph G obtained from K_n by deleting the edges in Ω, denoted by $K_n\backslash E(\Omega)$, is the graph of order n such that $xy \in E(G)$ if and only if xy is not an edge in any subgraph in Ω.

Theorem 3.3 *Let K_n be complete graph of order $n \geq 3$ and Ω be an independent family of complete proper subgraphs of K_n, each of order at least 2. If $G = K_n\backslash E(\Omega)$ and $k = \min\{p : K_p \in \Omega\}$, then*

$$g(G) = \begin{cases} 2, & \text{if } k = 2 \\ 3, & \text{if } k = 3 \\ 4, & \text{if } k \neq 2,3 \text{ and } |\Omega| \geq 2 \\ p, & \text{if } \Omega = \{K_p\} \end{cases}.$$

Proof: Consider the ff. cases:

Case 1. Suppose $k = 2$ and let $V(K_2) = \{a, b\}$. Then $d(a, b) = 2$. Since $x \in I[a, b]$ for all $x \notin V(K_2)$, it follows that $V(G) \setminus V(K_2) \subseteq I[a, b]$. This implies that $V(K_2)$ is a minimum geodetic set in G. Thus, $g(G) = 2$.

Case 2. Suppose $k = 3$. Let $V(K_3) = \{x, y, z\}$. Clearly, $V(G) \subseteq V(K_3) \subseteq I[V(K_3)]$; hence $V(K_3)$ is a geodetic set in G. If $g(G) = 2$, then there exists $S = \{a, b\}$ such that $d(a, b) = 2$ and $V(G) = I[S]$. In particular, $x \in I[a, b]$ for all $x \in V(G) \setminus S$. This implies that $K_2 = \langle\{a, b\}\rangle \in \Omega$, contrary to our assumption.

Case 3. Suppose $k \geq 4$ and $|\Omega| \geq 2$. It is routine to show that $g(G) \geq 4$. Let K_p, K_q be distinct elements of Ω, $a, b \in V(K_p)$ and $x, y \in V(K_q)$. Then $I[S] = V(G)$ for $S = \{a, b, x, y\}$. Thus, $g(G) = 4$.

Case 4. Suppose $\Omega = \{K_p\}$ and let S be a geodetic set in G. Let $x \in V(K_p)$. If $x \notin S$, then there exist $a, b \in S$ such that $x \in I[a, b]$. This implies that $d(a, b) = 2$; hence $a, b \in V(K_p)$. It follows that $ax, bx \notin E(G)$ which is impossible. Therefore, $x \in S$; hence, $V(K_p) \subseteq S$. Since $I[V(K_p)] = V(G)$, $V(K_p)$ is a minimum geodetic set. Therefore $g(G) = p$. □

The following is a quick consequence of Theorem 3.3.

Corollary 3.4 *Let K_n be complete graph of order $n \geq 3$. If G is the graph of order n obtained from K_n by deleting an independent family of edges, then $g(G) = 2$.*

The following result is due to Chartrand, Harary and Zang [10].

Theorem 3.5 *If v is a vertex of a graph G such that $\langle N(v)\rangle$ is complete (that is, $v \in V(G)$), then v belongs to every hull set and every geodetic set of G.*

Theorem 3.6 *Let K_n be complete graph of order $n \geq 3$ and Ω an independent family of complete proper subgraphs of K_n, each of order at least 2.*

If $G = K_n \backslash E(\Omega)$, *then*

$$h(G) = \begin{cases} 2 & \text{if } |\Omega| \geq 2 \\ p & \text{if } \Omega = \{K_p\}. \end{cases}$$

Proof: Suppose $G = K_n \backslash E(\Omega)$. Consider the following cases:

Case 1. Suppose $\Omega = \{K_p\}$. Let $S = V(K_p)$ and choose $x, y \in S$. Then $d_G(x, y) = 2$. If $z \notin S$, then $[x, z, y]$ is an x-y geodesic. Thus, $V(G) \subseteq I[S]$. This means that $[S] = V(G)$; that is, S is a hull set in G. Moreover, S is a minimum hull set in G by Theorem 3.5. Therefore, $h(G) = |S| = p$.

Case 2. Suppose $|\Omega| \geq 2$. Fix $H = K_q \in \Omega$. Choose $a, b \in V(H)$ and let $A = \{a, b\}$. Clearly, $d_G(a, b) = 2$. Also, if $c \notin V(H)$, then $[a, c, b]$ is an a-b geodesic. This implies that $(V(G) \backslash V(H)) \cup A \subseteq I[A]$. Choose $x, y \in K_p$, where $K_p \in (\Omega \backslash \{H\})$. Then $x, y \in I[A]$ and $d_G(x, y) = 2$. Note that if $w \in V(H)$, then $[x, w, y]$ is an x-y geodesic. This shows that $V(H) \subseteq I^2[A]$. Since $I[A] \subseteq I^2[A]$, it follows that $V(G) \subseteq I^2[A]$. Thus, $I^2[A] = V(G)$; that is, A is a hull set in G. Since singleton subsets are convex sets, it follows that A is a minimum hull set in G. Therefore $h(G) = 2$. □

The following result is a direct consequence of Theorem 3.6.

Corollary 3.7 *Let K_n be complete graph of order $n \geq 3$. If G is a graph of order n obtained from K_n by deleting an independent set of edges, then* $h(G) = 2$.

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