# On Semi-Open and Semi-Closed Functions

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**Abstract**

In this paper the concepts of semi-open and semi-closed functions are introduced. The conditions involved are weaker than those of the open and closed functions are introduced. Some equivalent statements of these newly-defined concepts are likewise offered.

**Keywords:** semi-open, semi-closed, function, semi-continuous, semi-closure

## 1 Introduction

In 1963, N. Levine [4] introduced the concept of semi-open set in topological spaces. He then used this new term to define the property of semi-continuity for functions from a topological space into another topological space. In [1], Canoy and Benitez gave some equivalent statements of semi-continuity. In particular, an equivalent statement involving the new concept called semi-closure of a set is given. This present paper deals with the semi-open and the semi-closed functions. We shall give some characterizations of these new properties of functions.

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Throughout this paper, X and Y are topological spaces.

**Definition 1.1** A subset $O$ of $X$ is *semi-open* if $O \subseteq cl[int(O)]$ (closure of the interior of $O$). Equivalently, $O$ is semi-open open if there exists an open set $G$ in $X$ such that $G \subseteq O \subseteq cl(G)$. A subset $F$ of $X$ is *semi-closed* if the complement $F^c$ of $F$ is semi-open.

It is easy to see from Definition 1.1 that every open set (closed set) in $X$ is semi-open (resp. semi-closed) in $X$. Although the family of all semi-open sets in a topological space $X$ does not form a topology on $X$, Levine (see [4]) managed to show the following useful result.

**Theorem 1.2 (Levine)** *Let $\{O_\alpha : \alpha \in I\}$ be a collection of semi-open sets in $X$. Then $\cup\{O_\alpha : \alpha \in I\}$ is a semi-open set in $X$.*

**Definition 1.3** A function $f : X \to Y$ is *semi-continuous* on $X$ if $f^{-1}(O)$ is semi-open in $X$ for every open set $O$ in $Y$.

Observe that by definition, a continuous function is always semi-continuous. However, there are semi-continuous functions which are not continuous. Readers are referred to [4] for an illustration.

**Definition 1.4** A function $f : X \to Y$ is *semi-open* (semi-closed) on $X$ if $f(G)$ is semi-open (resp. semi-closed) in $X$ for every open set $G$ in $X$.

The following two results are simple. The second of which says that for bijective functions, the two concepts are equivalent.

**Theorem 1.5** *If $f : X \to Y$ is open (closed) on $X$, then $f$ semi-open (resp. closed) on $X$.*

**Theorem 1.6** *Let $f : X \to Y$ be a bijective function. Then $f$ is semi-open if and only if it is semi-closed $X$.*

**Remark 1.7** *A semi-open function need not be semi-continuous.*

To see this, consider $X = Y = \{1,2,3,4\}$, $T_1 = \{\varnothing, X, \{3\}, \{2,4\}, \{2,3,4\}\}$, and $T_2 = \{\varnothing, X, \{2\}, \{3\}, \{2,3\}, \{2,4\}, \{2,3,4\}\}$. Define $f : (X, T_1) \to (Y, T_2)$ by $f(x) = x$. Since $T_1 \subset T_2$, it follows that $f$ is an open function. Thus, by Theorem 1.5, $f$ is a semi-open function. Now, take $O = \{2\}$. Then $O$ is open in $Y$ and $f^{-1}(O) = \{2\}$. Since $T_1 - int(\{2\}) = \varnothing$, it follows that $T_1 - cl[T_1 - int(\{2\})] = \varnothing$. Hence $\{2\} \not\subseteq T_1 - cl[T_1 - int(\{2\})]$, that is, $f^{-1}(O) = \{2\}$ is not semi-open in $X$. Accordingly, $f$ is not semi-continuous on $X$.

**Remark 1.8** *A semi-continuous function need not be semi-open.*

Consider $Y = \{1,2,3,4\}$ and $T = \{\varnothing, Y, \{3\}, \{2,4\}, \{2,3,4\}\}$. Let $A = \{1,4\}$ carry the relative topology $T_A$ and define $f : (A, T_A) \to (Y, T)$ by $f(a) = a$ ($f$ is really the inclusion function). Since $f^{-1}(O) = O \cap A$ is open in $A$ for every $O$ in $T$, it follows that $f$ is a continuous function. Therefore, $f$ is a semi-continuous function. Next, choose $G = \{4\}$. Since $G = A \cap 2,4$, $G$ is an open set in $A$. By definition, $f(G) = \{4\}$. Since $\{4\}$ is not in $T$, $T - int(\{4\}) = \varnothing$. This implies that $T - cl[T - int(\{4\})] = \varnothing$. Therefore $f(G) = \{4\}$ is not semi-open in $Y$. Accordingly, $f$ is not semi-open on $A$.

**Theorem 1.9** *Let $f : X \to Y$ be a function. Then the following statements are equivalent.*

(a) *$f$ is semi-open on $X$.*

(b) $f(int(A)) \subseteq cl[int(f(A))]$ *for every* $A \subseteq X$.

(c) $f(B)$ *is semi-open for every element* $B$ *of a basis for* $Y$.

(d) *For each* $p \in X$ *and every open set* $O$ *in* $X$ *containing* $p$*, there exists a semi-open set* $G$ *in* $Y$ *such that* $f(p) \in G \subseteq f(O)$.

*Proof*: (a):(b): Suppose $f$ is semi-open on $X$ and let $A$ be a subset of $X$. Since $f$ is semi-open and $int(A)$ is an open set in $X$, $f(int(A))$ is a semi-open set in $Y$ by Definition 1.4. By Definition 1.1, we have $f(int(A)) \subseteq cl[int(f(int(A))]$ (closure and interior are with respect to the topology on $Y$). Now, since $int(A) \subseteq A$, $f(int(A)) \subseteq f(A)$. Therefore $f(int(A)) \subseteq cl[int(f(int(A))] \subseteq cl[int(f((A))]$.

(b):(c): Assume that statement (b) holds. Let $B$ be an element of a base for $X$. Then $B$ is open, i.e., $int(B) = B$. Therefore, by our assumption,

$$f(int(B)) = f(B) \subseteq cl[int(f(B))].$$

This means that $f(B)$ is semi-open in $Y$ by Definition 1.1.

(c):(d): First, suppose that the image of every basic (open) set is semi-open. Let $p \in X$ and $O$ an open set in $X$ containing $p$. Then there exists a basic element $B$ such that $p \in B \subseteq O$. Next, set $G = f(B)$. Then by assumption, $G$ is semi-open set. Moreover, $f(p) \in G$ and $G \subseteq f(O)$.

(d):(a): Assume that statement in (d) holds. Let $O$ be an open set in $X$ and $z \in f(O)$. Choose $x \in O$ such that $f(x) = z$. By our assumption, there exists a semi-open set $G_z$ containing $z$ such that $G_z \subseteq f(O)$. It follows that $f(O) = \cup\{G_z : z \in f(O)\}$. Therefore, by Theorem 1.2, $f(O)$ is a semi-open set in $Y$. Therefore, since $O$ was arbitrary, $f$ is semi-open in $X$.

The proof of the theorem is complete. □

We shall now give a characterization of semi-closed functions. To do this, first we recall a definition and some results in [2].

**Definition 1.10** Let $A$ be a subset of $X$. A point $p \in X$ is a *semi-closure point* of $A$ if for every semi-open set $G$ in $X$, $p \in G$ implies that $G \cap A \neq \varnothing$. The set of all semi-closure points of $A$ is denoted by $scl(A)$.

**Theorem 1.11** *Let $A \subseteq X$. Then*

(a) $A \subseteq scl(A)$;

(b) *$A$ is semi-closed if and only if $A = scl(A)$; and*

(c) *$scl(A)$ is the smallest semi-closed set containing $A$.*

**Theorem 1.12** *A function $f : X \to Y$ is semi-closed on $X$ if and only if $scl(f(A)) \subseteq f(cl(A))$ for every subset $A$ of $X$.*

*Proof*: Let $A$ be a subset of $X$ and suppose that $f$ is semi-closed on $X$. Since $cl(A)$ is a closed set, the set $f(cl(A))$ is semi-closed in $Y$. Note that since $A \subseteq cl(A)$, we have $f(A) \subseteq f(cl(A))$. Theorem 1.11(c) now tells us that $scl(f(A)) \subseteq f(cl(A))$.

Conversely, suppose that $scl(f(A)) \subseteq f(cl(A))$ for every subset $A$ of $X$. Let $K$ be a closed subset of $X$. Then, by Theorem 1.11(a), $f(K) \subseteq scl(f(K))$. By our assumption, it follows that $f(K) \subseteq scl(f(K)) \subseteq f(cl(K))$. Since $K$ is closed, we know that $cl(K) = K$. Therefore we have $f(K) \subseteq scl(f(K)) \subseteq f(K)$ showing that $f(K) = scl(f(K))$. By Theorem 1.11(b), $f(K)$ is a semi-closed set in $Y$. Therefore $f$ is a semi-closed function. □

# References


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