On the Steiner Number of the Composition of Graphs

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Abstract

Given a connected graph G and a non-empty subset W of $V(G)$, a Steiner W-tree is a tree of minimum order that contains all of W. Let $S(W)$ denote the set of all vertices of G that lie on any Steiner W-tree. If $S(W) = V(G)$, then W is said to be a Steiner set of G. The Steiner number $st(G)$ of G is defined as the mimimum cardinality of a Steiner set of G. In this paper, we characterize the Steiner sets in the composition $G[H]$ of a non-trivial connected graph G and a disconnected graph H. We then present a formula that can be used to determine the Steiner number $st(G[H])$.

Keywords: graph, Steiner W-tree, Steiner set, Steiner number, composition

1 Introduction

Given a connected graph G and a non-empty subset W of $V(G)$, a Steiner W-tree is a tree of minimum order that contains all of W. Let $S(W)$ denote the set of all vertices of G that lie on any Steiner W-tree; the set $S(W)$ is

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referred to as the Steiner interval of W. If $S(W) = V(G)$, then W is said to be a Steiner set of G. Accordingly, a Steiner set of minimum cardinality is called a minimum Steiner set and this cardinality is the Steiner number $st(G)$ of G. Since every connected graph G contains a spanning tree, $V(G)$ is always a Steiner set of G. Therefore, if G is connected of order $n \geq 2$, then $2 \leq st(G) \leq n$.

Steiner sets and Steiner numbers have been studied recently in [2] and [4]. A more recent investigation is in [1], where the authors characterized the Steiner sets in the join $G + H$ and composition $G[H]$ of two non-trivial connected graphs G and H. One of the formulas obtained there can be stated as follows: $st(G[H]) = \min\{|V(H) \setminus S'| : S'$ is a cutset of H and no proper subset of S' disconnects $H\}$ if H is non-complete and G has a vertex of degree $|V(G)| - 1$; otherwise, $st(G[H]) = st(G) \cdot |V(H)|$. Although descriptions of Steiner sets of $G+H$ with either G or H (or both) disconnected can be found in [1], the equally challenging task of describing the Steiner sets of $G[H]$ with H disconnected has been postponed.

In this paper, we shall characterize the Steiner sets in the composition $G[H]$ of a non-trivial connected graph G and a disconnected graph H. Our main objective is to obtain a formula that can be used to determine the Steiner number $st(G[H])$ of the composition $G[H]$. (Note that graph-theoretic terms not specifically defined here may be found in [3].)

2 Results

The composition of two graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ whose elements satisfy the adjacency condition: (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$. Note that even if H is disconnected, the composition $G[H]$ is always connected provided G is connected.

Let $W \subseteq V(G[H])$. The *G-projection* W_G of W and the *H-projection* W_H of W are defined as follows:

$$\begin{aligned} W_G &= \{u : (u,v) \in W \text{ for some } v \in V(H)\}, \\ W_H &= \{v : (u,v) \in W \text{ for some } u \in V(G)\}. \end{aligned}$$

Theorem 2.1 *Let G be a non-trivial connected graph and H a disconnected graph. Let $W \subseteq V(G[H])$ such that $|W_G| = 1$. Then W is a Steiner set of $G[H]$ if and only if the following conditions hold:*

(i) $W_G = \{u\}$ for some $u \in V(G)$ with $deg_G(u) = |V(G)| - 1$;

(ii) $W_H = V(H)$.

Proof: Assume that W is a Steiner set of $G[H]$; let $W_G = \{u\}$ and let $a \in V(G)$ such that a is adjacent to u. Clearly, $W = \{u\} \times W_H$. Since W is a Steiner set of $G[H]$ and G is non-trivial, it follows that $\langle W \rangle$ is disconnected; hence, any Steiner W-tree must have at least $|W| + 1$ vertices. But the adjacency of the vertices in $V(G[H])$ immediately shows that for any $b \in V(H)$, the subgraph $\langle W \cup \{(a,b)\} \rangle$ is connected. Therefore, thinking of any spanning tree of $\langle W \cup \{(a,b)\} \rangle$, we can now conclude that every Steiner W-tree has exactly $|W| + 1$ vertices. Since W is a Steiner set of $G[H]$, by definition every vertex of $G[H]$ is in some Steiner W-tree. Consequently, u must be adjacent to all the other vertices of G, or $deg_G(u) = |V(G)| - 1$.

Next, we show that $W_H = V(H)$ by contradiction. Assume that W_H is a proper subset of $V(H)$. Since W is a Steiner set of $G[H]$, every vertex of $G[H]$ is in some Steiner W-tree (whose order is $|W|+1$); thus, the subgraph $\langle W \cup \{(u,y)\}\rangle$ is connected for every $y \in V(H) \setminus W_H$. Since $W = \{u\} \times W_H$, it follows that the subgraph $\langle W_H \cup \{y\}\rangle$ is connected in H for every $y \in V(H) \setminus W_H$. Consequently, $\langle W_H \cup (V(H) \setminus W_H)\rangle = \langle V(H)\rangle$ is connected, a contradiction to the hypothesis that H is disconnected. Therefore we must have $W_H = V(H)$.

The converse is straightforward. □

The following result is a consequence of the above theorem.

Corollary 2.2 *Let G be a non-trivial connected graph and H a disconnected graph. Then G has no spanning star subgraph if and only if $|W_G| \geq 2$ for every Steiner set W of $G[H]$.*

Proof: Suppose G has no spanning star subgraph. Suppose further that $G[H]$ has a Steiner set W with $|W_G| = 1$. Then $W_G = \{u\}$ for some $u \in V(G)$ with $deg(u) = m = |V(G)| - 1$. This implies that G has a spanning subgraph $K_{1,m}$, contrary to our assumption. Thus, $|W_G| \geq 2$ for every Steiner set W of $G[H]$.

Conversely, suppose $|W_G| \geq 2$ for every Steiner set W of $G[H]$. Assume, to the contrary that G has a spanning subgraph $K_{1,n-1}$, where $n = |V(G)|$. Let $K_{1,n-1} = \langle u\rangle + \overline{K}_{n-1}$, where $\overline{K}_{n-1}$ is the empty graph of order $n-1$, and $u \in V(G)$. Then $W^* = \{u\} \times V(H)$ is a Steiner set of $G[H]$, by Theorem 2.1. This clearly contradicts our assumption of the Steiner sets of $G[H]$. Therefore G does not have a spanning star subgraph. □

Lemma 2.3 *Let G be a non-trivial connected graph and H a disconnected graph. Let $W \subseteq V(G[H])$ such that $|W_G| \geq 2$, and let T^* be a Steiner W-tree. If T is a spanning tree of the subgraph $\langle (V(T^*))_G \rangle$, then T is a Steiner W_G-tree. Moreover, $|V(T^*)| = |W| + |V(T) \setminus W_G|$.*

Proof: The fact that T^* is connected in $G[H]$ implies that the subgraph $\langle (V(T^*))_G \rangle$ is connected in G, and hence has a spanning tree. Let T then be a spanning tree of $\langle (V(T^*))_G \rangle$. Since $V(T^*)$ contains W, it follows that $V(T)$ contains W_G. Furthermore, $|V(T^*)| \geq |W| + |V(T) \setminus W_G|$.

Assume for contradiction that T is not a Steiner W_G-tree. Then there exists a tree T' in G containing all of W_G such that $|V(T')| < |V(T)|$. If $V(T') = W_G$, then $\langle W_G \rangle$ is connected. Since $|W_G| \geq 2$, it follows that $\langle W \rangle$ is connected also. Consequently, $|W| = |V(T^*)|$ and $|W_G| = |V(T)|$. From $|V(T)| > |V(T')|$, we obtain a contradiction $|W_G| > |V(T')|$. Thus, $V(T') \setminus W_G$ is nonempty. Moreover, from the argument leading to the contradiction, the subgraph $\langle W \rangle$ must be disconnected and so is the subgraph $\langle W_G \rangle$ (note that W_G is not a singleton).

Let $R_1, R_2, \cdots$ be the vertex sets of the components of $\langle W_G \rangle$. We propose to show that we can form a tree in $G[H]$ containing W such that its order is smaller than that of T^*. To do this, consider all vertices of the form (x, z) where z is a fixed element of $V(H)$ and $x \in V(T') \setminus W_G$. If $\langle W \cap (R_i \times V(H)) \rangle$ is connected, take a spanning tree T_i^*. (Note that if $\langle W \cap (R_j \times V(H)) \rangle$ is disconnected, then necessarily R_j is a singleton.) By using the adjacency relation of the vertices of the tree T' in G, form a tree T^{**} in $G[H]$ in the following manner: connect each T_i^*, through one of its vertices, to any appropriate vertex (x, z), $x \in V(T') \setminus W_G$; if $\langle W \cap (R_j \times V(H)) \rangle$ is disconnected, disregard all its edges and then connect all its vertices to any appropriate vertex (x, z), $x \in V(T') \setminus W_G$. Furthermore, include in T^{**} any edge connecting (x, z) and (x', z) whenever $x, x' \in V(T') \setminus W_G$ and $xx' \in E(T')$. Now the tree T^{**} obviously contains W, and that $|V(T^{**})| = |W| + |V(T') \setminus W_G|$. Since $|V(T') \setminus W_G| < |V(T) \setminus W_G|$, it follows that $|V(T^{**}| < |V(T^*)|$, a contradiction to the hypothesis that T^* is a Steiner W-tree. This contradiction finally implies that T must be a Steiner W_G-tree.

Now let us consider the possibilities for W_G. If $W_G = V(T)$, then $\langle W_G \rangle$ is connected. Since $\langle W \rangle$ must be connected also, it follows that $V(T^*) = W$, and hence the equation $|V(T^*)| = |W| + |V(T) \setminus W_G|$ holds. On the other hand, if W_G is a proper subset of $V(T)$, then $\langle W_G \rangle$ is necessarily disconnected. Using a similar argument as in the preceding paragraph, we can form a tree T_Δ in $G[H]$ containing W and with exactly $|W| + |V(T) \setminus W_G|$ vertices. Combining this with the inequality in the first paragraph and the fact that T^* is a Steiner W-tree, we obtain $|V(T^*)| = |W| + |V(T) \setminus W_G|$. This completes the proof. □

Theorem 2.4 *Let G be a non-trivial connected graph and H a disconnected graph, and let $W \subseteq V(G[H])$ such that $|W_G| \geq 2$. If W is a Steiner set of $G[H]$, then the G-projection W_G of W is a Steiner set of G. Moreover, $W = W_G \times V(H)$.*

Proof: To show that $S(W_G) = V(G)$, let $u \in V(G)$. Let $v \in V(H)$ and let $x = (u, v)$. Since W is a Steiner set of $G[H]$, there exists a Steiner W-tree T^x containing x as a vertex. Now if T is a spanning tree of $\langle (V(T^x))_G \rangle$, then T is a Steiner W_G-tree by Lemma 2.3. Since $u \in V(T)$, it follows that every vertex of G is in some Steiner W_G-tree. Consequently, $V(G) \subseteq S(W_G)$. But the other inclusion $S(W_G) \subseteq V(G)$ is obvious. Therefore, $S(W_G) = V(G)$ and, hence, W_G is a Steiner set of G.

To show that $W = W_G \times V(H)$, it suffices to show that $W_G \times V(H) \subseteq W$. For contradiction, suppose there exists a vertex $(x, y) \in W_G \times V(H)$ such that $(x, y) \notin W$. Let T^* be a tree in $G[H]$ such that $W \cup \{(x, y)\} \subseteq V(T^*)$. If T' is a spanning tree of $\langle(\hat{V}(T^*))_G\rangle$, then $|V(T^*)| \geq |W| + 1 + |V(T') \setminus W_G|$. By Lemma 2.3, T^* cannot be a Steiner W-tree. Consequently, $(x, y) \notin S(W)$, a contradiction to the assumption that W is a Steiner set of $G[H]$. Hence, every vertex in $W_G \times V(H)$ is in W, or $W_G \times V(H) \subseteq W$. □

Theorem 2.5 *Let G be a nontrivial connected graph and H a disconnected graph, and let $W \subseteq V(G[H])$ such that $|W_G| \geq 2$. Then, W is a Steiner set of $G[H]$ if and only if $W = Q \times V(H)$, where Q is a Steiner set of G.*

Proof: Suppose W is a Steiner set of $G[H]$. If we take $Q = W_G$, then by Theorem 2.4, Q is a Steiner set of G and that $W = Q \times V(H)$.

Conversely, suppose $W = Q \times V(H)$ where Q is a Steiner set of G. If $Q = V(G)$, then obviously $Q \times V(H)$ is a Steiner set of $G[H]$. So assume that Q is a proper subset of $V(G)$. Necessarily, $\langle Q \rangle$ is disconnected. As a consequence, all Steiner Q-trees are of order $|Q| + k$ for some positive integer k. By Lemma 2.3, every Steiner W-tree has an order $(|V(H)| \cdot |Q|) + k$. Let $V(G) = \{u_1, u_2, \cdots, u_{|V(G)|}\}$ and $V(H) = \{v_1, v_2, \cdots, v_{|V(H)|}\}$. For an arbitrary element $(u_i, v_j) \in V(G[H])$, let T^{u_i} be a Steiner Q-tree containing u_i. Moreover, denote by $R_1, R_2, \cdots$ the vertex sets of the components of $\langle Q \rangle$. Clearly $\langle R_s \times V(H) \rangle$ is connected if and only if R_s is not a singleton. Now if R_α is not a singleton, take a spanning tree T_α of $\langle R_\alpha \times V(H) \rangle$ such that $(u_i, v_j) \in V(T_\alpha)$ in case $u_i \in R_\alpha$. By using the adjacency of the vertices of the tree T^{u_i} in G, form a tree $T^{(u_i,v_j)}$ in $G[H]$ in the following manner: connect each T_α, through one of its vertices, to any appropriate vertex (x, v_j), $x \in V(T^{u_i}) \setminus Q$; if R_β is a singleton, disregard all the edges of $\langle R_\beta \times V(H) \rangle$ and then connect all its vertices to any appropriate vertex (x, v_j), $x \in V(T^{u_i}) \setminus Q$. In addition, include in $T^{(u_i,v_j)}$ any edge connecting (x, v_j) and (x', v_j) whenever $x, x' \in V(T^{u_i}) \setminus Q$ and $xx' \in E(T^{u_i})$. The vertex set of the constructed tree $T^{(u_i,v_j)}$ has the following properties: $W \subseteq V(T^{(u_i,v_j)})$, $(u_i, v_j) \in V(T^{(u_i,v_j)})$ and $|V(T^{(u_i,v_j)})| = (|V(H)| \cdot |Q|) + k$. So $T^{(u_i,v_j)}$ must be a Steiner W-tree. Consequently, $(u_i, v_j) \in S(W)$, or $V(G[H]) \subseteq S(W)$. Since $S(W) \subseteq V(G[H])$, we have $S(W) = V(G[H])$. Therefore, W is a Steiner set of $G[H]$. □

Our final result is a consequence of Theorem 2.1, Corollary 2.2 and Theorem 2.5. This result gives the Steiner number of the composition $G[H]$, where G is nontrivial and connected while H is disconnected.

Theorem 2.6 *Let G be a nontrivial connected graph and H a disconnected graph. If G has a vertex of degree $|V(G)|-1$, then $st(G[H]) = |V(H)|$; otherwise, $st(G[H]) = st(G) \cdot |V(H)|$.*

Proof: Suppose G has a vertex of degree $|V(G)| - 1$. By Theorem 2.1, the Steiner sets of $G[H]$ whose G-projections are singletons are exactly those of the form $W = W_G \times V(H)$, where $W_G = \{u\}$ for some $u \in V(G)$ with $deg_G(u) = |V(H)| - 1$. On the other hand, by Theorem 2.5, the Steiner sets of $G[H]$ whose G-projections are not singletons are exactly those of the form $W = Q \times V(H)$, where Q is a Steiner set of G. As a consequence, we have $st(G[H]) = |V(H)|$.

Suppose now that G does not have a vertex of degree $|V(G)|-1$. Then by Corollary 2.2, the G-projections of the Steiner sets of $G[H]$ have cardinalities greater than one. So by applying Theorem 2.5, we obtain $st(G[H]) = st(G) \cdot |V(H)|$. □

We end this paper with a sample of specific situations where Theorem 2.6 can be applied. Note that by inspection the Steiner number of the path P_n, where $n \geq 2$, is 2, while the Steiner number of the cycle C_n is either 2 or 3, depending on whether n is even or odd.

Corollary 2.7 *Let H be a disconnected graph. Let K_n be the complete graph of order n; let F_n and W_n be the fan and wheel of order $n+1$, respectively. Also, let P_n and C_n be the path and cycle of order n, respectively. Then the following hold:*

(i) $st(K_n[H]) = |V(H)|$, *where* $n \geq 2$;

(ii) $st(F_n[H]) = |V(H)|$, *where* $n \geq 2$;

(iii) $st(W_n[H]) = |V(H)|$, *where* $n \geq 3$;

(iv) $st(P_n[H]) = 2 \cdot |V(H)|$, *where* $n \geq 2$;

(v) $st(C_n[H]) = r \cdot |V(H)|$, *where r is 2 or 3 depending on whether n is even or odd.*

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