On Mappings Satisfying the SC Condition

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Abstract

This study introduces a certain property that is satisfied by some functions. We show that this property is weaker than that of a weak continuous surjection. Since weak continuity is weaker than continuity, this property is weaker than that of continuous surjection. We shall show that under certain conditions this, and the interiority condition preserve local connectedness.

Keywords: mapping, weak continuity, SC condition, connectedness, interiority condition

1 Introduction

Chew and Tong defined in [1] the concept of weak continuity of a function. This new term must have been derived from the fact that the condition involved is strictly weaker than the condition for continuity of a function.

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One would easily recognize that the condition for weak continuity is obtained by slightly modifying the condition for continuity.

Another concept that was given and defined in [1] was the interiority condition of a function. It was shown in [1] that this condition, together with weak continuity, will imply continuity.

In this paper, we shall introduce a certain property that is satisfied by some functions and show that this new property is strictly weaker than that of a weak continuous surjection. We shall see that under certain conditions, this property and the interiority condition will preserve local connectedness.

Definition 1.1 A mapping $f : X \to Y$ is said to satisfy the SC condition if for every open set V in Y, $y \in V$ implies that $f^{-1}(y) \cap int\ f^{-1}(\overline{V}) \neq \varnothing$, where $f^{-1}(y) = \{x \in X : f(x) = y\}$.

Example 1.2 Let $X = \mathbb{R}$ be the set of real numbers with the usual topology U and $Y = \{a, b\}$ with the discrete topology D. Define the function $f : X \to Y$ by $f(x) = a$ if $x \leq 0$ and $f(x) = b$ if $x > 0$. Then f satisfies the SC condition.

Theorem 1.3 *If $f : X \to Y$ satisfies the SC condition, then it must be a surjection (an onto function).*

Proof: Let $y \in Y$. The SC condition implies that $f^{-1}(y) \cap int\ f^{-1}(\overline{V}) \neq \varnothing$. Since $int\ f^{-1}(\overline{V}) = int\ X = X$, it follows that there exists $x \in X$ such that $f(x) = y$. This proves that f is onto. □

Definition 1.4 A mapping $f : X \to Y$ is *weak continuous* at $x_0 \in X$ if for every open set V in Y containing $f(x_0)$, there exists an open set O in X

containing x_0 such that $f(O) \subseteq \overline{V}$. It is *weak continuous on* X if it is weak continuous at every point of X.

Example 1.5 Every continuous function $f : X \to Y$ is weakly continuous.

Theorem 1.6 *If $f : X \to Y$ is a weak continuous surjection, then f satisfies the SC condition.*

Proof: Suppose f is a weak continuous surjection. Let V be an open set in Y and $y \in V$. Since f is surjective, there exists an $x \in f^{-1}(V)$ such that $f(x) = y$. By weak continuity of f, there exists an open set O in X containing x such that $f(O) \subseteq \overline{V}$. Note that $f(O) \subseteq \overline{V}$ implies that $O \subseteq f^{-1}(f(O)) \subseteq f^{-1}(\overline{V})$. This shows that $x \in int\ f^{-1}(\overline{V})$. Since $x \in f^{-1}(y)$, it follows that $f^{-1}(y) \cap int\ f^{-1}(\overline{V}) \neq \varnothing$. Therefore f satisfies the SC condition. □

Remark 1.7 *The converse of Theorem 1.6 is not true.*

To see this, consider the function f defined in Example 1.2. Observe that for every open interval I containing $x = 0$, $f(I) = \{a, b\}$. Hence, for every open set O containing x we have $f(O) = \{a, b\}$. Now, $V = \{a\}$ is an open set containing $f(0) = a$. Since Y is the discrete space, $\overline{V} = V$. This implies that for every open set O containing x, $f(O) \not\subseteq \overline{V}$. Thus f is not weak continuous at $x = 0$.

Lemma 1.8 *Let $f : X \to Y$ be a bijection that satisfies the SC condition. If Y is a Hausdorff space, then f has closed point inverses, i.e., $f^{-1}(p)$ is closed in X for every $p \in Y$.*

Proof: Let $p \in Y$. Since f is onto, there exists an $x_0 \in X$ such that $f(x_0) = p$. Further, since f is one to one, $f^{-1}(p) = \{x_0\}$. Set $A = \{x \in X : f(x) \neq p\} = X \setminus \{x0\}$. We shall show that A is open. To this end, let $z \in A$. Then $f(z) \notin p$. Since Y is Hausdorff, there exists an open set V in Y such that $f(z) \in V$ but $p \notin \overline{V}$. By condition (SC), $f^{-1}(f(z)) \cap int\ f^{-1}(\overline{V}) = \{z\} \cap int\ f^{-1}(\overline{V}) \neq \varnothing$. This implies that there exists an open set O in X containing z such that $O \subseteq f^{-1}(\overline{V})$. Hence $f(O) \subseteq \overline{V}$. Note that if we can show that $O \subseteq A$, then we are done. So, suppose that $O \nsubseteq A$. Then $x_0 \in O$. Since $f(O) \subseteq \overline{V}$, it follows that $f(x_0) = p \in \overline{V}$, contrary to our choice of V. Thus, $O \subseteq A$ and A is open. Accordingly, $f^{-1}(p)$ is closed. □

The following result, which can be found in [3, p.139], gives a simple characterization of T_1-spaces.

Lemma 1.9 *A topological space X is a T_1-space if and only if every singleton subset $\{x\}$ of X is closed.*

Theorem 1.10 *Let $f : X \to Y$ be a bijection that satisfies the SC condition. If Y is a Hausdorff space, then X must be a T_1-space.*

Proof: Let $x \in X$ and put $p = f(x)$. Since f is one to one, $f^{-1}(p) = \{x\}$. By Lemma 1.8, the singleton set $\{x\}$ is closed. The desired result now follows from Lemma 1.9. □

Next, we shall show that for bijective functions, the SC condition and weak continuity are equivalent.

Theorem 1.11 *Let $f : X \to Y$ be a bijection. Then f satisfies the SC condition if and only if it is weak continuous on X.*

Proof: (:) Suppose f satisfies the SC condition. Let $x \in X$ and V be an open set in Y containing $f(x) = y$. By the (SC) condition, $f^{-1}(y) \cap int\, f^{-1}(\overline{V}) \neq \varnothing$. Since f is one to one, $f^{-1}(y) = \{x\}$. Thus, $x \in int\ f^{-1}(\overline{V})$, i.e., there exists an open set O in X containing x such that $O \subseteq f^{-1}(\overline{V})$. Therefore there exists an open set O in X containing x such that $f(O) \subseteq \overline{V}$. This shows that f is weak continuous at x. Since x was arbitrary, f is weak continuous on X.

($\Leftarrow$) This follows from Theorem 1.6. □

It is shown in [1] that a weak continuous function preserves connectedness. One might ask the question: "Is connectedness also preserved by functions satisfying the SC condition?" The function in Example 1.2 will show that the answer is "NO".

Next, we shall show that a connected open mapping satisfying a stronger condition than the (SC) condition preserves local connectedness.

Lemma 1.12 *If $f : X \to Y$ is a connected mapping and C is a component of Y, then $f^{-1}(C)$ is a union of some components of X.*

Proof: Let $x \in f^{-1}(C)$ and B_x the component of X containing x. Since f is connected and B_x is connected, $f(Bx)$ is connected. Thus, since $f(x) \in f(B_x) \cap C$, maximality of C implies that $f(B_x) \subseteq C$. It follows that $B_x \subseteq f^{-1}(C)$. This proves the assertion. □

Lemma 1.13 *If $f : X \to Y$ is a connected mapping, $A \subseteq X$ and $Q \subseteq Y$ such that $f(A) \subseteq Q$, then $g : A \to Q$, defined by $g(x) = f(x)$, is also connected.*

Proof: Let K be a connected subset of A. If K were disconnected in X, then there exist disjoint nonempty open sets H and G in X such that $K = (K \cap H) \cup (K \cap G)$. Set $E = A \cap H$ and $F = A \cap G$. Then E and F are disjoint nonempty open sets in A and $K = (K \cap E) \cup (K \cap F)$. This means that K is a connected set in A, contrary to our assumption. Thus, K is a connected subset of X. Since f is connected, it follows that $f(K) = g(K)$ is a connected set in Y and hence, in Q. This proves the lemma. □

Lemma 1.14 [2] *A space X is locally connected if and only if the components of each open set in X are open.*

Definition 1.15 A mapping $f : X \to Y$ is said to satisfy the *interiority condition* if for every open set V in Y, $int\ f^{-1}(\overline{V}) \subseteq f^{-1}(V)$. We say that f satisfies the (SCI) condition if it satisfies both the SC and the interiority conditions.

Example 1.16 The function f in Example 1.2 satisfies the interiority condition. Thus, f satisfies the SCI condition.

The following result is simple.

Lemma 1.17 *A mapping $f : X \to Y$ satisfies the interiority condition if and only if for every open set V in Y, $int\ f^{-1}(\overline{V}) = int\ f^{-1}(V)$.*

Proof: If f satisfies the interiority condition, then $int\ f^{-1}(\overline{V}) \subseteq f^{-1}(V)$ for every open set V in Y. Since $int\ f^{-1}(V)$ is the largest open set contained in $f^{-1}(V)$, it follows that $int\ f^{-1}(\overline{V}) \subseteq int\ f^{-1}(V)$. The inclusion $int\ f^{-1}(V) \subseteq int\ f^{-1}(\overline{V})$ is clear because $f^{-1}(V) \subseteq f^{-1}(\overline{V})$. Therefore, $int\ f^{-1}(\bar{}) = int\ f^{-1}(V)$ for every open set V in Y.

Conversely, if $int\ f^{-1}(\overline{V}) = int\ f^{-1}(V)$ for every open set V in Y, then $int\ f^{-1}(\overline{V}) \subseteq f^{-1}(V)$ because $int\ f^{-1}(V) \subseteq f^{-1}(V)$. Therefore, f satisfies the interiority condition. □

Theorem 1.18 *Let X be a locally connected space and Y a topological space. If $f : X \to Y$ is a connected open mapping satisfying the SCI condition, then Y is locally connected.*

Proof: Let V be an open set in Y and C a component of V. By Lemma 1.12 and Lemma 1.13, $f^{-1}(C)$ is a union of some components of $f^{-1}(V)$. Also, $f^{-1}(C) \cap int\ f^{-1}(V)$ is a union of some components of $int\ f^{-1}(V)$. To see this, let $x \in f^{-1}(C) \cap int\ f^{-1}(V)$ and D_x be the component of $int\ f^{-1}(V)$ containing x. Then D_x is a connected subset of $f^{-1}(V)$. Since $x \in f^{-1}(C)$, there exists a component B_x of $f^{-1}(V)$ such that $x \in B_x \subseteq f^{-1}(C)$. By the maximality of B_x, it follows that $D_x \subseteq B_x \subseteq f^{-1}(C)$. Hence, $D_x \subseteq f^{-1}(C) \cap int f - 1(V)$. This justifies our second statement.

Now, since $int\ f^{-1}(V)$ is open in X and X is locally connected, its components are open in X. Thus, $f^{-1}(C) \cap int\ f^{-1}(V)$ is open in X. We shall show that $C = f(f^{-1}(C) \cap int\ f^{-1}(V))$. To this end, let $y \in C$. Since $y \in V$ and f satisfies the (SC) condition, $f^{-1}(y) \cap int\ f^{-1}(\overline{V}) \neq \varnothing$. This means that there exists an $x \in f^{-1}(y) \cap int\ f^{-1}(\overline{V})$. Interiority property of

f and the fact that $f^{-1}(y) \subseteq f^{-1}(C)$ imply that $x \in f^{-1}(C) \cap int\ f^{-1}(V)$. Therefore, $y = f(x)$ is an element of $f(f^{-1}(C) \cap int\ f^{-1}(V))$. This shows that $C \subseteq f(f^{-1}(C) \cap int\ f^{-1}(V))$. Since $C \subseteq V$, it follows that $f(f^{-1}(C) \cap int\ f^{-1}(V)) \subseteq C \cap V \subseteq C$. Therefore, $C = f(f^{-1}(C) \cap int\ f^{-1}(V))$. Equality implies that C is open. □

References

[1] Chew, J., and Tong, J., *Some Remarks on Weak Continuity*, The American Mathematical Monthly, (1991) 931-934.

[2] Dugundji, J., *Topology*, New Delhi, Prentice Hall of India Private Ltd., 1975.

[3] Lipschutz, S., *General Topology*, Schaum's Outline Series, McGraw-Hill International Book Co., Singapore, 1981.