Lower Bounds for Bandwidth: Sharpness and Goodness

SEVERINO V. GERVACIO

Abstract

Let G be a graph of order n. How can the vertices of G be labeled bijec-tively by 1, 2, ..., n such that the maximum difference between labels of adjacent vertices is a minimum? This minimax value is called the bandwidth or density of the graph G, denoted by B(G). This problem was posed by Harary in 1963.

It is known that each of $\left\lceil \frac{n-1}{d(G)} \right\rceil, \left\lceil \frac{n}{\alpha(G)} \right\rceil - 1, \quad \chi(G) - 1$ *is a lower bound for B(G), where d(G), α(G), χ(G) are the diameter, independence number and chromatic number of G, respectively.*

Let κ(G) and β(G) denote the connectivity and the dominance number, respectively, of the graph G. We show that κ(G) and $\left\lceil \frac{n-\beta(G)}{2\beta(G)} \right\rceil$ *are also lower bounds for B(G).*

Each of the aforecited lower bounds is sharp in the sense that some graphs have bandwidth exactly equal to the lower bound. We show, however, that none of the lower bounds is generally a good lower bound. Specifically, we show that if B_i *and* B_j *are two of the five lower bounds, then there exist graphs G and H such that* $B_i(G) < B_j(G)$ *and* $B_i(H) > B_j(H)$, *except that* $B_2(G) \le B_3(G)$ *for all graphs G.*

1. Introduction

There are essentially two different forms of the bandwidth problem.
(1) For a graph G the problem is to label the n vertices v_i of G with distinct integers $f(v_i)$ from $\{1, 2, ..., n\}$ so that the maximum value of the difference

$$|f(v_i) - f(v_j)|,$$

taken over all adjacent pairs of vertices, is a minimum.
(2) For a real symmetric matrix M the problem is to find a symmetric permutation M' of M so that the maximum value of $|i - j|$ taken over all nonzero entries m'_{ij} is a minimum.

SEVERINO V. GERVACIO was a former Dean and former Vice Chancellor for Research and Extension, MSU-Iligan Institute of Technology. Currently a Professor of mathematics at De la Salle University, Gervacio has published several important papers on Graph Theory in local and international journals.

In the bandwidth problem for matrices, if we replace each nonzero entry of M by 1, then the result may be interpreted as the adjacency matrix of a graph. Thus, the two bandwidth problems are equivalent.

The bandwidth problem for matrices seems to have originated in the 1950's when structural engineers first analyzed steel frameworks by computer manipulation of their structural matrices. In order that operations like inversion and finding determinants of matrices take as little time as possible, attempts were made to discover an equivalent matrix in which all the nonzero entries lay within a narrow band about the main diagonal - hence the term "bandwidth."

The bandwidth problem for graphs, on the other hand, originated from the Jet Propulsion Laboratory (JPL) at Pasadena in 1962. Single errors in a 6-bit picture code were represented by edge differences in a hypercube whose vertices were words of the code. At JPL, L. H. Harper and A. W. Hales sought codes which minimized the maximum absolute error and the average absolute error. Thus were born the bandwidth and the bandwidth sum problems - at least for the cube. F. Harary publicized the problem in a conference in Prague.

Since the mid-sixties there has been a growing interest in the bandwidth problem for graphs. I encountered this problem first in 1977 when I was teaching graph theory for the first time, here in Ateneo.

In 1976, C. H. Papadimitriou proved that the graph bandwidth problem is NP-complete. (The bandwidth problem for matrices is therefore NP-complete also.)

2. Preliminary Concepts

By a **graph** G we understand a pair $\langle V(G),E(G)\rangle$, where $V(G)$ is a finite non-empty set of elements called **vertices**, and $E(G)$ is a set of 2-subsets of $V(G)$ called **edges**. For simplicity, an edge $e = \{a, b\} \in E(G)$ will be written as ab. We say that a and b are **adjacent** and we call a and b the **end-vertices** of the edge e. The numbers $|V(G)|$ and $|E(G)|$ are called the **order** and **size** of G, respectively.

The **distance** between two vertices x and y in a graph G, denoted by $d(x,y)$, is defined as the length of any shortest path in G with end-vertices x and y. The **diameter** of G, denoted by $d(G)$, is defined by $d(G) = \max\{d(a,b) : a, b \in V(G)\}$.

A set $S \subseteq V(G)$ is called an **independent set** in G if $xy \notin E(G)$ for all $x, y \in S$. The **independence number** of G, denoted by $\alpha(G)$, is defined to be the cardinality of a largest independent set in G.

The **chromatic number** of G, denoted by $\chi(G)$, is the minimum number of colors the vertices of G may be given such that adjacent vertices get different colors. Thus, $\chi(G)$ is the minimum number of independent subsets into which $V(G)$ may be partitioned.

The **connectivity** of G, denoted by $\kappa(G)$, is the minimum number of vertices needed to be removed from G to disconnect G or reduce G to an isolated vertex.

The **dominance number** of G, denoted by $\beta(G)$, is the minimum cardinality of a set of vertices $S \subseteq V(G)$ such that every vertex $x \notin S$ is adjacent to at least one vertex in S.

3. Some Graphs, Operations and Notations

The following definitions and notations are needed for an understanding of the graphs given in Table 1.

The **sum** of two (disjoint) graphs G and H, denoted by $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G) \text{ and } y \in V(H)\}$.

The **complement** of a graph G, denoted by $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ and where $xy \in E(\overline{G})$ if and only if $x \neq y$ and $xy \notin E(G)$.

The **complete graph of order *n***, denoted by K_n, is the graph of order n where xy is an edge for all distinct $x, y \in V(K_n)$. The **complete bipartite graph**, denoted by $K_{m,n}$ is the graph $\overline{K_m} + \overline{K_n}$.

The **path of order *n***, denoted by P_n, is the graph with vertices x_1, x_2, ..., x_n and with edges x_i, x_{i+1} $i = 1, 2, ..., n - 1$. The symbol $x_1x_2 ... x_n$ also denotes the path of order n. The cycle of order $n \geq 3$ is obtained from the path P_n by adding the edge x_nx_1.

Let K_m and K_n be two disjoint complete graphs. The graph $K_m | K_n$ is the graph obtained from K_m and K_n by adding an edge ab, where a is in $V(K_m)$ and b is in $V(K_n)$.

The **crown of order 2*n***, denoted by C_n, is formed by taking a cycle C_n with vertices x_i and another set of n independent vertices y_i and adding the edges x_iy_i, $i = 1, 2, ..., n$. If we take another cycle of or-der n with vertices z_i and add the edges z_iy_i, for $i = 1, 2, ..., n$ then the resulting graph is called the **double crown of order 3n**, denoted by $\mathbf{C'}_n$.

The **fan of order n+1**, denoted by F_n, is defined to be the graph $P_n + K_1$. The **double fan of order n+2** is the graph $P_n + K_2$. The **bifan**, denoted by $F_{m,n}$, is the graph $G + K_1$, where G consists of two disjoint paths P_m and P_n. The **twin fan of order 2n+2**, denoted by F'_n, is the graph obtained from two disjoint fans $x_1x_2...x_n + K_1$ and $y_1y_2...y_n + K_1$ by adding the edges x_1y_1 and x_ny_n.

Let $G_1, G_2, \ldots, G_n$ be disjoint graphs, each having a complete subgraph K_p. The ***K_p*-gluing** of the graphs G_i is the graph obtained by identifying the K_p's of the G_i's (one from each graph, in any manner). The K_p-gluing of t copies of K_n is denoted by $K_n(t,p)$.

The **Cartesian product** of two graphs G and H, denoted by $G \times H$, is the graph with $V(G \times H) = V(G) \times V(H)$ and where two vertices (a,b) and (c,d) are adjacent if and only if (i) $ac \in E(G)$ and $b = d$ or (ii) $a = c$ and $bd \in E(H)$.

The **wheel of order n+1**, denoted by W_n, is the graph $C_n + K_1$. The **biwheel**, denoted by $W_{m,n}$ is the graph $G + K_1$, where G consists of two disjoint cycles C_m and C_n.

If x is a real number, we denote by $\lfloor x \rfloor$ the greatest integer not exceeding x and by $\lceil x \rceil$ the smallest integer not less than x.

4. Main Results

An **integer label** λ of a graph G of order n is a bijective mapping $\lambda : V(G) \to \{1, 2, \ldots, n\}$. The **bandwidth** of G, denoted by $B(G)$, is defined by

$$B(G) = \min_{\lambda} \max_{ab \in E(G)} \{ |\lambda(a) - \lambda(b)| \},$$

where λ ranges over all integer labels of G.

It is known that $\left\lceil \frac{n-1}{d(G)} \right\rceil$, $\left\lceil \frac{n}{\alpha(G)} \right\rceil - 1$, $\chi(G) - 1$, are lower bounds for $B(G)$. These lower bounds were obtained in [3] from the observation that a graph G of order n is a spanning subgraph of $P_n^{B(G)}$, the graph obtained from P_n by adding the edge xy whenever the distance x and y in P_n does not exceed $B(G)$. In general, if k is a positive integer, G^k is the graph obtained from G by adding the edge xy whenever $d(x,y) \le k$, where $d(x,y)$ is the distance between x and y in G.

We first give and prove two new lower bounds for bandwidth.

Theorem 1. *$B(G) \geq \kappa(G)$ for any graph G.*

Proof. Let G be a graph of order n. Since G is a spanning subgraph of $P_n^{B(G)}$, it follows that $\kappa(G) \leq \kappa(P_n^{B(G)}) = B(G)$. □

Lemma 1. Let a, b, c be non-zero integers, and let x be any real number. (i) If $a \geq \left\lceil \frac{b}{c} \right\rceil$, then $c \geq \left\lceil \frac{b}{a} \right\rceil$; (ii) $\left\lceil \frac{\lceil x \rceil}{a} \right\rceil = \left\lceil \frac{x}{a} \right\rceil$.

Proof. (i) Suppose that $c < \left\lceil \frac{b}{a} \right\rceil$. Then $c < \left\lceil \frac{b}{\lceil b/c \rceil} \right\rceil \leq \left\lceil \frac{b}{b/c} \right\rceil =$ c, which is a contradiction.

(ii) Let $x = n - \varepsilon$, $0 \leq \varepsilon < 1$. In case $\varepsilon = 0$, (ii) clearly holds. So, let $0 < \varepsilon < 1$. Also, let $n = aq + r$, $0 \leq r < a$. Then

$$\left\lceil \frac{\lceil x \rceil}{a} \right\rceil = \left\lceil \frac{n}{a} \right\rceil = \left\lceil \frac{aq+r}{a} \right\rceil = q + \left\lceil \frac{r}{a} \right\rceil.$$

On the other hand,

$$\left\lceil \frac{x}{a} \right\rceil = \left\lceil \frac{aq+r-\varepsilon}{a} \right\rceil = q + \left\lceil \frac{r-e}{a} \right\rceil = q + \left\lceil \frac{r}{\varepsilon} \right\rceil,$$

since $0 < \varepsilon < 1$ and $0 \leq r < a$. □

Lemma 2. Let $1 \leq k < n$. Then $\beta(P_n^k) = \left\lceil \frac{n}{2k+1} \right\rceil$.

Proof. Consider the graph P_n^k, where P_n is the path with vertices $1, 2, \ldots, n$ and edges $\{i, i+1\}$, $i = 1, 2, \ldots, n-1$. Let $n = (2k+1)q + r$, $0 < r < 2k+1$. If $r = 0$, then it is easy to verify that the vertices $k+1$, $3k+2$, ..., $(2q-1)k+q$, form a (unique) minimum dominating set in P_n^k. Hence, $B(P_n^k) = q = \left\lceil \frac{n}{2k+1} \right\rceil$. If $r \neq 0$, then the vertices $k+1$, $3k+2$, ..., $(2q-1)k+q$, n form a minimum dominating set in P_n^k. Hence, it follows that $B(P_n^k) = q + 1 = \left\lceil \frac{n}{2k+1} \right\rceil$. □

Theorem 2. If G is a graph of order n, then $B(G) \geq \left\lceil \frac{n-\beta(G)}{2\beta(G)} \right\rceil$.

Proof. As in the proof of Theorem 1, we have $\beta(G) \geq \beta(P_n^{B(G)})$ since $G \subseteq P_n^{B(G)}$. But $\beta(P_n^{B(G)}) = \left\lceil \frac{n}{2k+1} \right\rceil$ by Lemma 2, where $k = B(G)$. By Lemma 1(i), $2B(G) + 1 \geq \left\lceil \frac{n}{\beta(G)} \right\rceil$. Therefore, $2B(G) \geq \left\lceil \frac{n-\beta(G)}{\beta(G)} \right\rceil$. This implies that $B(G) \geq \frac{1}{2}\left\lceil \frac{n-\beta(G)}{\beta(G)} \right\rceil$. Hence, $B(G) \geq \left\lceil \frac{1}{2}\left\lceil \frac{n-\beta(G)}{\beta(G)} \right\rceil \right\rceil$. By Lemma 1(ii), we finally get $B(G) \geq \left\lceil \frac{n-\beta(G)}{2\beta(G)} \right\rceil$. □

We shall denote by B_1, B_2, B_3, B_4, B_5 the quantities $\left\lceil \frac{n-1}{d(G)} \right\rceil$, $\left\lceil \frac{n}{\alpha(G)} \right\rceil - 1$, $\chi(G) - 1$, $\kappa(G)$ and $\left\lceil \frac{n-\beta(G)}{2\beta(G)} \right\rceil$, respectively.

We now show that each of the bounds B is sharp, i.e., $B(G_i) = B_i$ for some graph G_i. It is not difficult to check that if $1 \leq k < n$, then $B(P_n^k) = k = B_i$, for $i = 1, 2, 3, 4, 5$. This proves the sharpness of each of the five lower bounds B_i.

Although each lower bound B_i is sharp, none of them is better than the rest in the following sense: If B_i and B_j are any two among the five lower bounds, then there exist graphs G and H such that $B_i(G) < B_j(G)$ and $B_i(H) > B_j(H)$, except that $B_3(G) \geq B_2(G)$ for all graphs G.

To prove the foregoing statement, let $\chi(G) = k$ and let $V_1, V_2, \ldots, V_n$ form a partition of $V(G)$ into k independent sets. Then $|V_i| \leq \alpha(G)$ for each i and hence $k\alpha(G) \geq |V(G)| = n$. This implies that $k \geq \frac{n}{\alpha(G)}$. Since $k = \chi(G)$ is an integer, it follows that $\chi(G) > \left\lceil \frac{n}{\alpha(G)} \right\rceil$. Consequently, $B_3(G) > B_2(G)$. □

Table 1 below gives examples of graphs for which $B_i < B_j$, $B_i > B_j$, and $B_i = B_j$.

Table 1. Comparing the lower bounds B_i

i	j	$B_i > B_j$		$B_i < B_j$		$B_i = B_j$*
1	2	$K_{m,n}$,	$m \geq n \geq 2$	$K_m \| K_n$	$m \geq 2, n \geq 2, m+n \geq 9$	p odd
1	3	F_n,	$n \geq 5$	$P_m + K_n$,	$3 \leq m \leq n+3$	$p = 1$ or 2
1	4	Tree T,	not a path	$P_m + P_n$,	$m \geq 2, n \geq 2, \|m-n\| \leq 1$	$n = \lceil 3p/2 \rceil$
1	5	C_n,	$n \geq 3$	F'_n,	$n \geq 11$	all values
2	3	not possible		W_n,	$n \geq 4$	$p = 1$
2	4	$K_n(t,p)$	$n > 2p+1-\lceil p/t \rceil$	$K_n(t,p)$	$n < 2p+1-\lceil p/t \rceil$	$n = 1 + \lfloor 3p/2 \rfloor$
2	5	$C_m \times K_n$,	$n \geq 3$	$W_{m,n}$,	$m+n \geq 7$	p odd
3	4	$F_{m,n}$,	$m+n \geq 3$	$C_m \times P_n$,	$m \geq 3, n \geq 2$	$p = n - 1$
3	5	$\sum_{i=1}^{t} \overline{K_n}$,	$t \geq 6, n \leq 3$	$P_n + \overline{K_2}$,	$n \geq 9$	$p = 1$ or 2
4	5	C'_n,	$n \geq 3$	$C_n + \overline{K_2}$,	$n \geq 17$	$n = \lceil 3p/2 \rceil$

*Each graph in this column is $K_n(2,p)$, $1 \leq p < n$.

References

[1] Bernaldez, J. M. and Gervacio, S. V., *Research topics in graph theory and combinatorics*, a research project funded by NRCP, 1990.

[2] Gervacio, S. V., *Minimizing the maximum difference between integer labels of adjacent vertices in a graph*, in: **Proc. of the Annual Sci. Meeting of the Nat. Acad. of Sci. and Tech.**, Manila, 1989.

[3] Gervacio, S. V., *Minimizing the maximum difference between integer labels of adjacent vertices in a graph* - II, **Discrete Math**, 1991.

[4] Gervacio, S. V., *On the bandwidth of the planar grid*, in: **Proc. of the Annual Conv. of the Math. Soc. of the Phil.**, 1991.

[5] Gervacio, S. V. and Imasa, C. N., *Minimizing the maximum difference between integer labels of adjacent vertices in a graph* - III, unpublished paper, 1991.

[6] Harary, F., *A graph labeling problem*, **Graph Theory and Its Applications,** Smolenice Symp., Prague, 1964.

[7] Imasa, C. N., *On the density of graphs*, Master's Thesis, MSU-Iligan Inst. of Tech., 1991-1992.