Path Chromatic Index of the Composition of Some Special Graphs

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Abstract

Path chromatic index is one of the many coloring invariants of graphs. H.C. Serate [6] had presented an in-depth study of such coloring invariant. However, none has been established in the path chromatic index of the composition of graphs. Hence, the goal of this paper is to determine the path chromatic index of some special graphs.

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1 Introduction

Problems on graph coloring invariant have been undertaken by a number of graph theorists. The origin of all graph coloring problems is the famous

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Four Colour Problem posted in 1852. There had been several studies on partitioning the edge-set as well as the vertex-set of graphs. Among these studies is that of H.C. Serate's [6] *On the Path Chromatic Index of Some Graphs*. In her study, the path chromatic index of path, cycle, star, complete graph; and, other special graphs were established.

In this paper, we are faced with the problem of investigating the path chromatic index of the composition of some special graphs. Specifically, we aim to determine the path chromatic index of $K_1(G)$, $G(K_1)$, $\overline{K}_n(G)$, $P_m(\overline{K}_n)$, $P_m(P_n)$ and $P_m(G)$, where G is any graph.

2 Preliminaries

A *path* of length $n-1$, denoted by P_n, is a sequence $[x_1, x_2, \ldots, x_n]$ of distinct vertices of G where $[x_i, x_{i+1}]$ is an edge of G for all $i = 1, 2, \ldots, n-1$. The vertices x_1 and x_n are called *end-vertices* of the path. A *linear forest* is a graph whose components are paths.

Two graphs G and H are *isomorphic* if there exists a one-to-one correspondence between their vertex sets, which preserve adjacency. The *composition* of two graphs $G_1 = (X_1, E_1)$ and $G_2 = (X_2, E_2)$ is the graph $G_1(G_2)$ where $V(G_1(G_2)) = X_1 \times X_2$ and $[(x_1, x_2), (y_1, y_2)] \in E(G_1(G_2))$ if and only if (1) $x_1 = y_1$ and $[x_2, y_2] \in E_2$ or (2) $[x_1, y_1] \in E_1$.

The *path chromatic index* of a graph G, denoted by $\chi'_\infty(G)$ is the minimum number of colors the edges of G can be given so that each monochromatic color class induces a linear forest. Equivalently, it is the minimum number of subsets into which $E(G)$ can be partitioned into subsets $E_1, E_2, \ldots, E_n$ so that each $\langle E_i \rangle$ is a linear forest.

The following results are in [6].

Theorem 2.1 *Let* $G = P_n$*, the path of order* $n \geq 2$*. Then* $\chi'_\infty(G) = 1$

Theorem 2.2 *Let* $G = C_n$*, the cycle of order* $n \geq 3$*. Then* $\chi'_\infty(G) = 2$

Theorem 2.3 *Let* G *be the totally isolated graph* $\overline{K}_n$*. Then* $\chi'_\infty(G) = 0$.

Remark 2.4 $K_1(G) \cong G(K_1) \cong G$ *for any graph* G.

Remark 2.5 $\overline{K}_n(G) \cong G_1 \cup G_2 \cup \ldots \cup G_n$ *for any* G*, where* $G \cong G_1 \cong G_2 \cong \ldots \cong G_n$ *and* G_i*'s are disconnected.*

3 Main Results

Theorem 3.1 *If* $\chi'_\infty(G) = t$*, then*

$$\chi'_\infty(K_1(G)) = \chi'_\infty(G(K_1)) = t,$$

for any graph G.

Proof: Let G be any graph with $\chi'_\infty(G) = t$. From Remark 2.4, $K_1(G) \cong G(K_1) \cong G$. Thus $\chi'_\infty(K_1(G)) = \chi'_\infty(G(K_1)) = t$. □

Theorem 3.2 *If* $\chi'_\infty(G) = r$*, then* $\chi'_\infty(\overline{K}_n(G)) = r$ *for all* N *and for any graph* G*, where* $\overline{K}_n$ *is the totally isolated graph of order* n.

Proof: Suppose G is any graph and $\chi'_\infty(G) = r$. Observe that $\overline{K}_n(G)$ is just the union of n graphs G which are disconnected to one another [Remark 2.5]. We color each G the same with that of r colors.

Hence $\chi'_\infty(\overline{K}_n(G)) = r$. □

Theorem 3.3 *$\chi'_\infty(P_m(\overline{K}_n)) = n$, for $m \geq 3$ and for any n, where P_m is the path of order m and $\overline{K}_n$ is the totally isolated graph of order n.*

Proof: Label the vertices of $P_M(\overline{K}_n)$ as shown in the Figure 1 below.

$P_M(\overline{K}_n)$:

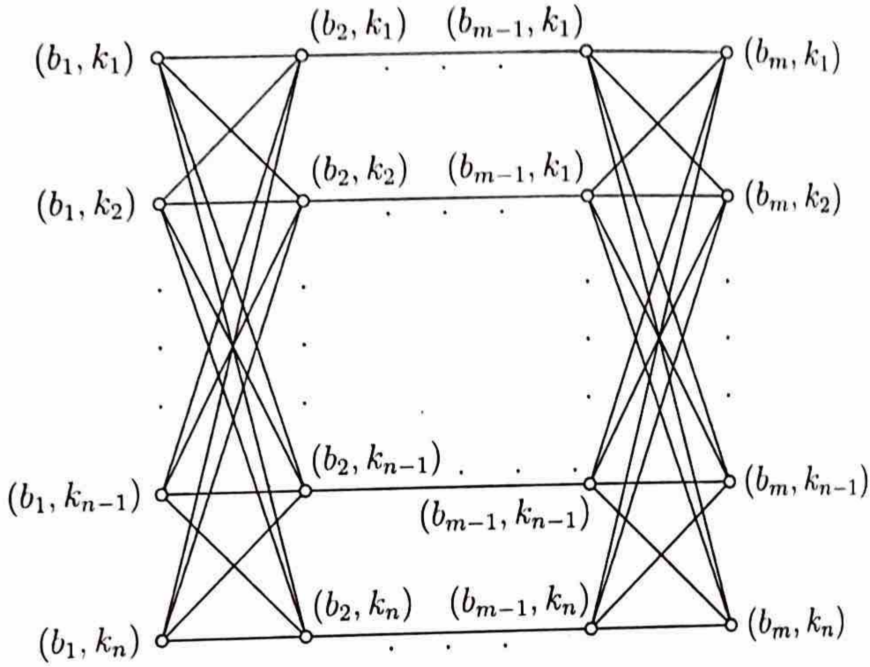

Figure 1:

Observe that the maximum degree of $P_M(\overline{K}_n)$ is $2n$. At most two edges incident to the vertex with maximum degree of $2n$ can be given the same color. Hence n colors will be the minimum number of color all edges incident to it. Thus $\chi'_\infty(P_m(\overline{K}_n)) \geq n$. Now, consider the following cases.

Case 1. $m < n$. We partition the linear forest of $P_m(\overline{K}_n)$ into subsets $L_1, L_2, \ldots, L_n$ such that $L_i \cap L_j = \emptyset$, where $i, j = 1, 2, \ldots, n$. Let

$$\begin{aligned} L_1 &= \{[(b_1,k_1),(b_2,k_1),\ldots,(b_m,k_1)],[(b_1,k_2),(b_2,k_2),\ldots,(b_m,k_2)],\ldots,\\ &\quad [(b_1,k_n),(b_2,k_n),\ldots,(b_m,k_n)]\} \\ L_2 &= \{[(b_1,k_1),(b_2,k_2),\ldots,(b_m,k_m)],[(b_1,k_2),(b_2,k_3),\ldots,(b_m,k_{m-1})],\ldots,\\ &\quad [(b_1,k_{n-1}),(b_2,k_n),\ldots,(b_m,k_{m-2})]\} \end{aligned}$$

$$\vdots$$

$$\begin{aligned} L_n &= \{[(b_1,k_1),(b_2,k_n),\ldots,(b_m,k_{n-(m-2)})],[(b_1,k_2),(b_2,k_1),\\ &\quad \ldots,(b_m,k_{n-(m-3)})],\ldots,[(b_1,k_n),(b_2,k_{n-1}),\ldots,(b_m,k_{n-(m-1)})]\}. \end{aligned}$$

Hence $L_i \cap L_j = \varnothing$ and $P_m(\overline{K}_n) = \bigcup_{i=1}^{n} L_i$. Color each L_i with a single color. The minimum number of colors to color $P_m(\overline{K}_n$ is n because if we color L_n with any of the colors used in $L_1, L_2, \ldots, L_{n-1}$, at least one cycle is induced. Hence $\chi'_\infty(P_m(\overline{K}_n)) = n$.

Case 2. $m = n$. We partition the linear forest of $P_m(\overline{K}_n)$ into n subsets $M_1, M_2, \ldots, M_n$ such that $M_i \cap M_j = \emptyset$, where $i, j = 1, 2, \ldots, n$. Let

$$\begin{aligned} M_1 &= \{[(b_1,k_1),(b_2,k_1),\ldots,(b_m,k_1)],[(b_1,k_2),(b_2,k_2),\ldots,(b_m,k_2)],\ldots,\\ &\quad [(b_1,k_n),(b_2,k_n),\ldots,(b_m,k_n)]\}\\ M_2 &= \{[(b_1,k_1),(b_2,k_2),\ldots,(b_m,k_n)],[(b_1,k_2),(b_2,k_3),\ldots,(b_m,k_1)],\ldots,\\ &\quad [(b_1,k_n),(b_2,k_1),\ldots,(b_m,k_{n-1})]\}\\ &\vdots\\ M_n &= \{[(b_1,k_1),(b_2,k_n),\ldots,(b_m,k_2)],[(b_1,k_2),(b_2,k_1),\ldots,(b_m,k_3)],\ldots,\\ &\quad [(b_1,k_n),(b_2,k_{n-1}),\ldots,(b_m,k_1)]\}. \end{aligned}$$

Thus $M_i \cap M_j = \emptyset$ and $P_m(\overline{K}_n) = \bigcup_{i=1}^{n} M_i$. Color each M_i with a single color. The minimum numbers of colors to color $P_m(\overline{K}_n)$ is n since if we color M_n with any of the colors used in $M_1, M_2, \ldots, M_n$ at least one cycle is induced. Hence $\chi'_\infty(P_m(\overline{K}_n)) = n$.

Case 3. $m > n$, where $m = n + q$. Again, we partition the linear forest of $P_m(\overline{K}_n)$ into n subsets $N_1, N_2, \ldots, N_n$ such that $N_i \cap N_j = \emptyset$, where

$i, j = 1, 2, \ldots, n$. Let

$$\begin{aligned}
N_1 &= \{[(b_1, k_1), (b_2, k_1), \ldots, (b_m, k_1)], [(b_1, k_2), (b_2, k_2), \ldots, (b_m, k_2)], \ldots, \\
&\quad [(b_1, k_n), (b_2, k_n), \ldots, (b_m, k_n)]\} \\
N_2 &= \{[(b_1, k_1), (b_2, k_2), \ldots, (b_n, k_n), (b_{n+1}, k_1), \ldots, (b_{n+q}, k_q)], \\
&\quad [(b_1, k_2), (b_2, k_3), \ldots, (b_n, k_1), (b_{n+1}, k_2), \ldots, (b_{n+q}, k_{q+1})], \ldots, \\
&\quad [(b_1, k_n), (b_2, k_1), \ldots, (b_n, k_{n-1}), (b_{n+1}, k_n), \ldots, (b_{n+q}, k_{q+(n-1)})]\} \\
&\vdots \\
N_n &= \{[(b_1, k_1), (b_2, k_n), \ldots, (b_n, k_2), (b_{n+1}, k_1), (b_{n+2}, k_n), \ldots, (b_{n+q}, k_{n-(q-3)})], \\
&\quad [(b_1, k_2), (b_2, k_1), \ldots, (b_n, k_3), (b_{n+1}, k_2), (b_{n+2}, k_n), \ldots, (b_{n+q}, k_{n-(q-3)})], \ldots \\
&\quad [(b_1, k_n), (b_2, k_{n-1}), \ldots, (b_n, k_1), (b_{n+1}, k_n), \ldots, (b_{n+q}, k_{n-(q-1)})]\}
\end{aligned}$$

Hence $N_i \cap N_j = \emptyset$ and $P_m(\overline{K}_n) = \bigcup_{i=1}^{n} N_i$. Color each N_i with a single color. The minimum numbers of colors to color $P_m(\overline{K}_n)$ is n since if we color M_n with any of the colors used in $N_1, N_2, \ldots, N_n$ at least one cycle is induced. Hence $\chi'_\infty(P_m(\overline{K}_n)) = n$

Therefore, in any case, $\chi'_\infty(P_m(\overline{K}_n)) = n$. □

Theorem 3.4 $\chi'_\infty(P_m(P_n)) = n + 1$, *for* $m \geq 3$ *and* $n \geq 2$, *where* P_m *and* P_n *are paths of order* m *and* n, *respectively.*

Proof: Using the figure above, let

$$\begin{aligned}
L &= \{[(b_1, k_1), (b_1, k_2), \ldots, (b_1, k_n)], [(b_2, k_1), (b_2, k_2), \ldots, (b_2, k_n)], \ldots, \\
&\quad [(b_m, k_1), (b_m, k_2, \ldots, (b_m, k_n)]\}
\end{aligned}$$

which induces a linear forest. Observe that $P_m(P_n) = P_m(\overline{K}_n) \cup L$. Therefore, from Theorem 3.3, $\chi'_\infty(P_m(P_n)) = \chi'_\infty(P_m(\overline{K}_n)) + \chi'_\infty(L) = n + 1$. □

Remark 3.5 *For $n = 1$, the theorem above does not hold.*

Theorem 3.6 $\chi'_\infty(P_m(C_n)) = n + 2$ *for $m \geq 3$ and $n \geq 3$, where P_m is the path of order m and C_n is the cycle of order n.*

Proof: Using the figure above, let

$$M = \{[(b_1, k_1), (b_1, k_n)], [(b_2, k_1), (b_2, k_n)], \ldots, [(b_m, k_1), (b_m, k_n)]\}$$

Note that $P_m(C_n) = P_m(p_n) \cup M$. From Theorem 3.4, $\chi'_\infty(P_m(C_n)) = \chi'_\infty(P_m(P_n)) + \chi'_\infty(m) = n + m$. □

Remark 3.7 *Let G be any graph of order n and P_m be the path of order $m \geq 3$. Then $P_m(G) \cong P_m(\overline{K}_n) \cup G_1 \cup G_2 \cup \ldots \cup G_m$, where $G \cong G_1 \cup G_2 \cup \ldots \cup G_m$ and G_i's are disconnected to one another.*

As an immediate consequence of the above remark and the preceding theorems, we the the following.

Corollary 3.8 $\chi'_\infty(P_m(G)) = \chi'_\infty(P_m(\overline{K}_n)) + \chi'_\infty(G) = n + 1$.

Proof: This follows from Remark 3.7. □

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