The Monotone and the Dominated Convergence Theorems for an Integral Involving a System of Paths

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Abstract

Recently, Fu introduced the concept of a system of paths and used this to define an integral. For a system consisting of neighborhoods, this integral is equivalent to the Henstock integral. This paper gives the Monotone and the Dominated Convergence theorems of the E-path integral defined by Fu in [3].

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1 Preliminary Concepts and Known Results

In what follows, we define some basic concepts and state some of the known results we shall need in the suceeding section. Interested readers may see [1]

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and [3] for the proofs of these known results and for a better understanding of the integral.

Definition 1.1 Let $\mathbb{R}$ be the real line. A *path* leading to x is a set $E_x \subset \Re$ such that $x \in E_x$ and x is a point of accumulation of E_x. A *system of paths* is a collection $E = \{E_x : x \in \mathbb{R}\}$ such that each E_x is a path leading to x. A system of paths $E = \{E_x : x \in \mathbb{R}\}$ is said to be *bilateral* if for every point $x \in \mathbb{R}$ and every $\epsilon > 0$, $(x - \epsilon, x) \cap E_x \neq \varnothing$ and $(x, x + \epsilon) \cap E_x \neq \varnothing$. It is said to satisfy the *intersection condition* if there is associated with E a positive function δ on $\mathbb{R}$ such that if $0 < y - x < \min\{\delta(x), \delta(y)\}$, then $E_x \cap E_y \cap [x, y] \neq \varnothing$.

Definition 1.2 Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths. A family C of closed subintervals of $[a, b]$ is an *E-full cover* of $[a, b]$ if there is a positive function δ on $[a, b]$ so that every interval $[y, z]$ of $[a, b]$ for which $y, z \in E_x, y \leq x \leq z$ and $0 < z - y < \delta(x)$ necessarily belongs to the collection C.

For convenience, we shall refer to the function δ in Definition 1.2 as the positive function corresponding to the E-full cover C.

The following results are proved in [3].

Lemma 1.3 *Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths and $\delta(x) > 0$ be a positive function on $[a, b]$. Then the collection $C = \{[u, v] \subset [a, b] : u, v \in E_x, u \leq x \leq v, 0 < v - u < \delta(x)$ for some $x \in [a, b]\}$ is an E-full cover of $[a, b]$.*

Lemma 1.4 (Intersection Lemma) *Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths. If C_1 and C_2 are E-full covers of $[a, b]$, then $C_1 \cap C_2$ is also an E-full cover of $[a, b]$.*

Definition 1.5 Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths and C an E-full cover of $[a, b]$. A tagged division $D = \{([u, v]; x)\}$ of $[a, b]$ is called a *C-partition* of $[a, b]$ if each $[u, v]$ belongs to C, $u, v \in E_x$ and $u \leq x \leq v$ for every interval-point pair $([u, v]; x)$ in D.

The following result guarantees the existence of a C-partition of an interval for every given E-full cover C of $[a, b]$. See [1] and [3] for its proof.

Theorem 1.6 (Thomson's Lemma) *Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths which is bilateral and has the intersection condition. If C is an E-full cover of $[a, b]$, then there is a C-partition of any interval $[a, b]$.*

In what follows, $E = \{E_x : x \in \mathbb{R}\}$ is a fixed system of paths that is bilateral, and satisfies the intersection condition.

Definition 1.7 A real-valued function f defined on $[a, b]$ is E-path integrable to the number A if for every $\epsilon > 0$ there exists an E-full cover C of $[a, b]$ such that for any C-partition $D = \{([u, v]; \xi)\}$ of $[a, b]$,

$$|(D) \sum f(\xi)(v - u) - A| < \epsilon.$$

The E-path integral A of f, if it exists, is unique. In symbols we write

$$(EP) \int_a^b f(t)dt = (EP) \int_a^b f = A.$$

It is shown in [1] and [3] that the set of all E-path integrable real-valued functions defined on $[a, b]$ is a real vector space and that E-path integrability over the whole interval implies E-path integrability on every subinterval. The following results were also proved.

Theorem 1.8 *If $f, g : [a, b] \to \mathbb{R}$ are E-path integrable on $[a, b]$ and $f(x) \leq g(x)$ for almost all $x \in [a, b]$, then*

$$(EP) \int_a^b f(t)dt \leq (EP) \int_a^b g(t)dt.$$

Theorem 1.9 *If $f(x) = 0$ almost everywhere in $[a, b]$, then f is E-path integrable to zero on $[a, b]$.*

Theorem 1.10 (Henstock's Lemma) *If $f : [a, b] \to \mathbb{R}$ is E-path integrable on $[a, b]$ with E-primitive F defined by $F(x) = \int_a^b f(t)dt$ for every $x \in [a, b]$, then for every $\epsilon > 0$, there exists an E-full cover C of $[a, b]$ such that for any C-partition $D = \{([u, v]; \xi)\}$ of $[a, b]$,*

$$(D) \sum |f(\xi)(v - u) - F(v) + F(u)| < \epsilon.$$

Definition 1.11 A function $F : [a, b] \to \mathbb{R}$ is said to be of *bounded variation* if $V(F; [a, b]) = sup\{(D) \sum |F(v) - F(u)|\}$ is finite, where supremum is taken over all divisions $D = \{([u, v]; \xi)\}$ of $[a, b]$.

2 Results

The following result is known as the Uniform Convergence Theorem.

Theorem 2.1 (Unifom Convergence Theorem) *If $\{f_n\}$ is a sequence of real-valued E-path integrable functions defined on $[a, b]$ that converges uniformly to f on $[a, b]$, then f is E-path integrable on $[a, b]$ and*

$$\lim_{n \to \infty} (EP) \int_a^b f_n = (EP) \int_a^b f.$$

Proof: Let $\epsilon > 0$. By uniform convergence, there exists a natural number N such that for all $m, n \geq N$ and for all $x \in [a, b]$,

$$|f_n(x) - f_m(x)| < \epsilon. \tag{1}$$

By hypothesis, there exist E-full covers C_n and C_m of $[a, b]$ such that if $D_1 = \{([u, v]; \xi)\}$ is a C_n-partition of $[a, b]$, and $D_2 = \{([u, v]; \xi)\}$ is a C_m-partition of $[a, b]$, then

$$|(D_1) \sum f_n(\xi)(v - u) - (EP) \int_a^b f_n| < \epsilon$$

and

$$|(D_2) \sum f_n(\xi)(v - u) - (EP) \int_a^b f_m| < \epsilon. \tag{2}$$

Let $n, m \geq N$ and put $C = C_n \cap C_m$. By Lemma 1.4, C is an E-full cover of $[a, b]$. Let $D = \{([u, v]; \xi)\}$ be a C-partition of $[a, b]$. Then D is both a C_n- and a C_m-partition of $[a, b]$. Consequently, by (1) and (2), we have

$$\begin{aligned} |(EP) \int_a^b f_n - (EP) \int_a^b f_m| &\leq |(EP) \int_a^b f_n - (D) \sum f_n(\xi)(v - u)| \\ &\quad + (D) \sum |f_n(\xi) - f_m(\xi)|(v - u) \\ &\quad + |(D) \sum f_m(\xi)(v - u) - \int_a^b f_m| \\ &< (2 + b - a)\epsilon. \end{aligned} \tag{3}$$

This means that the sequence $\{(EP) \int_a^b f_n\}$ is a Cauchy sequence in $\mathbb{R}$ and thus converges to a real number, say A. Next, let $\epsilon > 0$. Then there is a natural number N such that

$$|(EP) \int_a^b f_N - A| < \epsilon \tag{4}$$

and

$$|f(x) - f_N(x)| < \epsilon \tag{5}$$

for all $x \in [a, b]$. Since f_N is E-path integrable on $[a, b]$, there is an E-full cover C_N of $[a, b]$ such that for any C_N-partition $D = \{([u, v]; \xi)\}$ of $[a, b]$,

$$|(D) \sum f_N(\xi)(v - u) - (EP) \int_a^b f_N| < \epsilon. \tag{6}$$

Let $C = C_N$. If $D = \{([u, v]; \xi)\}$ is a C-partition of $[a, b]$, then by (4), (5), and (6), we have

$$\begin{aligned}
|(D) \sum f(\xi) - A| &\leq (D) \sum |f(\xi) - f_N(\xi)|(v - u) \\
&\quad + |(D) \sum f_N(\xi)(v - u) - (EP) \int_a^b f_N| \\
&\quad + |(EP) \int_a^b f_N - A| \\
&< (2 + b - a)\epsilon.
\end{aligned}$$

This proves the theorem. □

Lemma 2.2 *Let $f : [a, b] \to \mathbb{R}$ be E-path integrable on $[a, b$ and let $F(x) = (EP) \int_a^x f(t)dt$ for each $x \in [a, b]$. If F is of bounded variation on $[a, b]$, then $|f|$ is E-path integrable and*

$$(EP) \int_a^b |f|(t)dt = V(F; [a, b]).$$

Proof: Let $\epsilon > 0$. Since F is of bounded variation on $[a, b]$, there exists a division $D_0 = \{[a_i, b_i] : 1 \leq i \leq n\}$ of $[a, b]$ such that

$$V(F; [a, b]) - \epsilon < \sum_{i=1}^{p} |F(b_i) - F(a_i)|. \tag{7}$$

For each $x \in [a, b]$, let

$$\begin{aligned} \delta_1(x) &= dist(x, S \setminus \{x\}) \\ &= inf\{|x - y| : y \in S \setminus \{x\}\} , \end{aligned}$$

where $S = \cup_{i=1}^{n}\{a_i, b_i\}$ (the union of all endpoints of the intervals in D_0) Then for each $x \in [a, b]$, $\delta_1(x) > 0$.

Now, since f is E-path integrable on $[a, b]$, by the Henstock lemma, there exists an E-full cover C_1 of $[a, b]$ such that for any C_1-partition $D = \{([u, v]; \xi)\}$ of $[a, b]$, we have

$$(D) \sum |f(\xi)(v - u) - F(v) + F(u)| < \epsilon \tag{8}$$

Let $\delta^*(x)$ be the positive function corresponding to the E-full cover C_1 of $[a, b]$. For each $x \in [a, b]$, define

$$\delta(x) = \frac{1}{2} min\{\delta_1(x), \delta^*(x)\}$$

Let $C = \{[u, v] \subset [a, b] : u, v \in E_x, u \leq x \leq v, 0 < v - u < \delta(x)$ for some $x \in [a, b]\}$. Then C is an E-full cover of $[a, b]$. Suppose $D = \{([u, v]; \xi)\}$ is a C-partition of $[a, b]$. By definition of $\delta(x)$, D is a C_1-partition of $[a, b]$. Also, each interval $[u, v]$ in D is subset of some $[a_i, b_i]$. Hence, by (7),

$$\begin{aligned} V(F; [a, b]) - \epsilon &< \sum_{i=1}^{p} |F(b_i) - F(a_i)| \\ &\leq (D) \sum |F(v) - F(u)| \\ &\leq V(F; [a, b]). \end{aligned} \tag{9}$$

Therefore, by (8) and (9),

$$
\begin{aligned}
|(D)\sum |f|(\xi)(v-u) - V(F;[a,b]) \\
&\leq (D)\sum [|f|(\xi)(v-u) - |F(v)-F(u)|]| \\
&\quad +|(D)\sum |F(v)-F(u)| - V(F;[a,b])| \\
&\leq (D)\sum |[|f|(\xi)(v-u) - |F(v)-F(u)|]| + \epsilon \\
&\leq (D)\sum |f(\xi)(v-u) - F(v) + F(u)| + \epsilon \\
&< 2\epsilon\ .
\end{aligned}
$$

This proves the theorem. □

Theorem 2.3 *If $f, g : [a,b] \to \mathbb{R}$ are E-path integrable on $[a,b]$ and $|f| \leq g$, then $|f|$ is E-path integrable on $[a,b]$.*

Proof: Let F be the E-primitive of f defined by

$$F(x) = (EP)\int_a^x f(t)dt$$

for each $x \in [a,b]$. By Theorem 1.8 and the hypothesis, for every subinterval $[u,v]$ of $[a,b]$, we have

$$
\begin{aligned}
|F(v) - F(u)| &= |(EP)\int_u^v f(t)dt| \\
&\leq (EP)\int_u^v g(t)dt \\
&< \infty
\end{aligned}
$$

Thus

$$V(F;[a,b]) \leq \int_a^b g(t)dt < \infty\ ,$$

i.e., F is of bounded variation on $[a,b]$. By Lemma 2.2, it follows that $|f|$ is E-path integrable on $[a,b]$.

Theorem 2.4 *Let $f, g :\rightarrow \mathbb{R}$ be E-path integrable on $[a, b]$. If there exist E-path integrable functions h and $h*$ on $[a, b]$ such that $h* \leq f, g \leq h$, then $\max(f, g) = f \vee g$ and $\min\,(f, g) = f \wedge g$ are E-path integrable on $[a, b]$.*

Proof: For functions f and g we have

$$f \vee g = \frac{1}{2}(f + g + |f - g|)$$

and

$$f \wedge g = \frac{1}{2}(f + g - |f - g)|). \tag{10}$$

Hence

$$|f - g| = 2(f \vee g) - (f + g) \leq 2h - (f + g).$$

Since the set of all E-path integrable functions on $[a, b]$ is a real vector space, it follows that $f - g$ and $2h - (f + g)$ are E-path integrable on $[a, b]$. By Theorem 2.3, it follows that $|f - g|$ is E-path integrable on $[a, b]$. Thus by (10), $f \vee g$ is E-path integrable on $[a, b]$.

Similarly, $f \wedge g$ is E-path integrable on $[a, b]$. □

By induction on the number of functions involved, we obtain an extension of the above theorem.

Corollary 2.5 *Let $f_j :\rightarrow \mathbb{R}$, $j = 1, 2, \ldots, n$, be E-path integrable functions on $[a, b]$. Suppose that there exist E-path integrable functions h and g on $[a, b]$ such that $h \leq f_j \leq g$ for each $j = 1, 2, \ldots, n$. Then $\vee f_{j=1}^{n}$ and $\wedge f_{j=1}^{n}$ are E-path integrable on $[a, b]$.*

Theorem 2.6 (Monotone Convergence Theorem) *Let $\{f_n\}$ be an increasing sequence of E-path integrable real-valued functions defined on $[a, b]$*

and suppose that the sequence converges pointwise to a real-valued function f on $[a,b]$. Then $\{(EP)\int_a^b f_n\}$ converges to a real number if and only if f is E-path integrable on $[a,b]$. Moreover,

$$\lim_{n\to\infty}(EP)\int_a^b f_n = (EP)\int_a^b f.$$

Proof: Suppose that f is E-path integrable on $[a,b]$. Then, by hypothesis and Theorem 1.8, we have

$$(EP)\int_a^b f_n \le (EP)\int_a^b f_{n+1} \le (EP)\int_a^b f$$

for all n. Hence the sequence $\{(EP)\int_a^b f_n\}$ is increasing and is bounded. Thus it converges to a real number.

Conversely, suppose $\{(EP)\int_a^b f_n\}$ converges, say

$$A = \lim_{n\to\infty}(EP)\int_a^b f_n.$$

For each n, let

$$F_n(x) = (EP)\int_a^x f_n(t)dt$$

Then, given $\epsilon > 0$, there exists a natural number N_0 such that for all $k \ge N_0$, we have

$$|(EP)\int_a^b f_k - A| = A - F_k(b) < \epsilon. \tag{11}$$

Since $\{f_n\}$ converges pointwise to f, it follows that for each $\xi \in [a,b]$, there is a natural number $m(\epsilon,\xi) \ge N_0$ such that

$$|f_{m(\epsilon,\xi)}(\xi) - f(\xi)| < \epsilon. \tag{12}$$

By the Henstock lemma for E-path integrals, there exists an E-full cover C_n of $[a,b]$ for each natural number n such that for any C_n-partition $D =$

$\{([u, v]; \xi)\}$ of $[a, b]$,

$$(D)\sum |f_n(\xi)(v-u) - F_n(v) + F(v)| < \frac{\epsilon}{2^n}. \tag{13}$$

For each n, let $\delta_n(x)$ be the positive function corresponding to the E-full cover C_n of $[a, b]$. For each $\xi \in [a, b]$, define $\delta(x) = \delta_{m(\epsilon,\xi)}(\xi)$. Let

$$C = \{[u, v] \subset [a, b] : u, v \in E_x, u \le x \le v, 0 < v - u < \delta(x)\ .$$

Then C is an E-full cover of $[a, b]$. Let $D = \{([u, v]; \xi)\}$ be a C-partition of $[a, b]$. Then

$$\begin{aligned} |(D)\sum f(\xi)(v-u) - A| &\le (D)\sum |f(\xi) - f_{m(\epsilon,\xi)}(\xi)|(v-u) \\ &\quad +(D)\sum |f_{m(\epsilon,\xi)}(\xi) - F_{m(\epsilon,\xi)}(v) + F_{m(\epsilon,\xi)}(u)| \\ &\quad +|(D)\sum F_{m(\epsilon,\xi)}(v) - F_{m(\epsilon,\xi)}(u) - A|\ . \end{aligned}$$

Now, for every interval $[u, v]$ in D, the sequence

$$\{F_n([u, v])\} = \{F_n(v) - F_n(u)\}$$

is increasing and bounded. Hence it converges to a real number, say $F([u, v]) = F(v) - F(u)$. Since $F_n(b) = (D)\sum(F_n(v) - F_n(u))$, it follows that

$$\begin{aligned} A &= \lim_{n\to\infty} F_n(b) \\ &= (D)\sum(F(v) - F(u)) \\ &= F(b) - F(a)\ . \end{aligned}$$

Next, let $p = min\{m(\epsilon, \xi) : \xi$ is an associated point in $D\}$. Then $p \le m(\epsilon, \xi)$ for each ξ in D and $p \ge N_0$. Thus, by (11), we have

$$
\begin{aligned}
|(D)\sum(F_{m(\epsilon,\xi)}(v)-F_{m(\epsilon,\xi)}(u))-A| &= A-(D)\sum(F_{m(\epsilon,\xi)}(v)-F_{m(\epsilon,\xi)}(u)) \\
&\le A-(D)\sum(F_p(v)-F_p(u)) \\
&= A-F_p(b) \\
&< \epsilon. \qquad (14)
\end{aligned}
$$

Therefore, by (12), (13), and (14), we have

$$
\begin{aligned}
|(D)\sum f(\xi)(v-u)-A| &< \epsilon(b-a)+\epsilon+\epsilon \\
&= (b-a+2)\epsilon\ .
\end{aligned}
$$

This completes the proof of the theorem. □

Theorem 2.7 (Dominated Convergence Theorem) *Let $\{f_n\}$ be a sequence of E-path integrable real-valued functions defined on $[a,b]$ and suppose that $\{f_n\}$ converges pointwise to a real-valued function f on $[a,b]$. If there exist E-path integrable functions h and g on $[a,b]$ such that $g \le f_n \le h$ for all n, then f is E-path integrable on $[a,b]$ and*

$$\lim_{n\to\infty}(EP)\int_a^b f_n=(EP)\int_a^b f.$$

Proof: By Corollary 2.5 $\varphi_{j,k}=\vee_{n=j}^{k}(f_n)$ is E-path integrable for each pair (j,k) of positive integers, where $j<k$. For a fixed j, the sequence $\{\varphi_{j,k}\}_k$ is increasing and the corresponding sequence $\{(EP)\int_a^b\varphi_{j,k}\}_k$ of E-path integrals is increasing and bounded and so, convergent. By Theorem 2.6, the limit function $\sup\{f_n : n\ge j\}$ is E-path integrable on $[a,b]$.

Similarly, we can show that $\inf\{f_n : n \geq j\}$ is E-path integrable on $[a, b]$. Then

$$
\begin{aligned}
(EP)\int_a^b (inf\{f_n : n \geq j\}) &\leq inf\{(EP)\int_a^b f_n : n \geq j\} \\
&\leq sup\{(EP)\int_a^b f_n : n \geq j\} \\
&\leq (EP)\int_a^b (sup\{f_n : n \geq j\}). \qquad (15)
\end{aligned}
$$

Since $\{f_n\}$ converges pointwise to f, it follows that

$$\lim_{j\to\infty}(inf\{f_n(x) : n \geq j\}) = \lim_{j\to\infty}(sup\{f_n(x) : n \geq j\}).$$

By Theorem 2.6 applied to the sequences $\{inf\{f_n : n \geq j\}\}_j$ and $\{-sup\{f_n : n \geq j\}\}_j$, it follows that f is E-path integrable on $[a, b]$ and

$$
\begin{aligned}
\lim_{j\to\infty}(EP)\int_a^b (inf\{f_n(x) : n \geq j\}) &= (EP)\int_a^b f \\
&= \lim_{j\to\infty}(EP)\int_a^b (sup\{f_n(x) : n \geq j\}). \qquad (16)
\end{aligned}
$$

This proves the theorem. □

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