Embedding a Graph in a Harmonious Graph or a Graceful Graph

SEVERINO V. GERVACIO

Special names are given to graphs that admit vertex- or edge-labelings which satisfy some nice properties [1]. For instance, we have the so-called permutation graphs, full graphs, Fibonacci graphs, geometric graphs, magic graphs, graceful graphs, harmonious graphs, cordial graphs and strongly c-elegant graphs. A class of graphs called felicitous graphs generalizes both the harmonious graphs and the strongly c-elegant graphs. Lee, Schmeichel and Shee [2] gave several examples of felicitous graphs as well as examples of nonfelicitous graphs. They proved that every graph is a subgraph of a harmonious graph. Since every harmonious graph is felicitous, it follows then that every graph is a subgraph of a felicitous graph. We shall also prove that every graph is a subgraph of a graceful graph.

Definitions

By a **graph** G we mean a pair $G = \langle V(G), E(G)\rangle$, where $V(G)$ is a nonempty finite set of elements called **vertices** and $E(G)$ is a set of unordered pairs xy called **edges**, where x and y are distinct vertices in $V(G)$.

Let G be a graph with m edges. A **harmonious labeling** of G is a one-to-one mapping $\varphi : V(G) \rightarrow \{0, 1, 2, ..., m-1\}$ such that the induced mapping $\varphi^* : E(G) \rightarrow \{0, 1, 2, ..., m-1\}$ defined by $\varphi^*(e) = \varphi(x) + \varphi(y) \pmod{m}$ for all $e = xy \in E(G)$ is bijective. A graph is said to be **harmonious** if it admits a harmonious labeling.

Let G be a graph with m edges. A **felicitous labeling** [3] of G is a one-to-one mapping $\varphi : V(G) \rightarrow \{0, 1, 2, ..., m\}$ such that the induced mapping $\varphi^* : E(G) \rightarrow \{0, 1, 2, ..., m-1\}$ defined by $\varphi^*(e) = \varphi(x) + \varphi(y)$ (mod m) for all $e = xy \in E(G)$ is bijective. A graph is said to be **felicitous** if it admits a felicitous labeling. Clearly, every harmonious graph is a felicitous graph.

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The symbol K_n denotes the complete graph with n vertices such that $xy \in E(K_n)$ for all distinct vertices $x, y \in V(K_n)$. Graham and Sloane [2] proved that K_n is harmonious if and only if $n \leq 4$. Lee, Schmeichel and Shee [2] proved that K_n is felicitous if and only if $n \leq 4$. Figure 1 shows harmonious (and felicitous) labelings of K_n, $n \leq 4$.

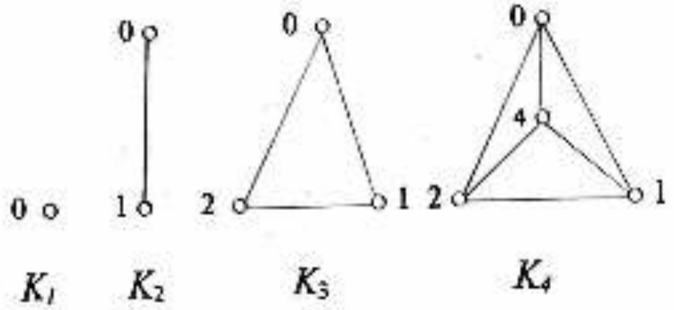

Fig. 1. Harmonious labelings of K_1, K_2, K_3, and K_4.

Let G be a graph with m edges. A **graceful labeling** of G is a one-to-one mapping $\varphi : V(G) \to \{0, 1, 2, \ldots, m\}$ such that the induced mapping $\varphi^* : E(G) \to \{1, 2, 3, \ldots, m\}$ defined by $\varphi^*(e) = |\varphi(x) - \varphi(y)|$ for all $e = xy \in E(G)$ is bijective. A graph is said to be **graceful** if it admits a graceful labeling.

Figure 2 shows the star S_6, the path P_5, and corresponding graceful labelings.

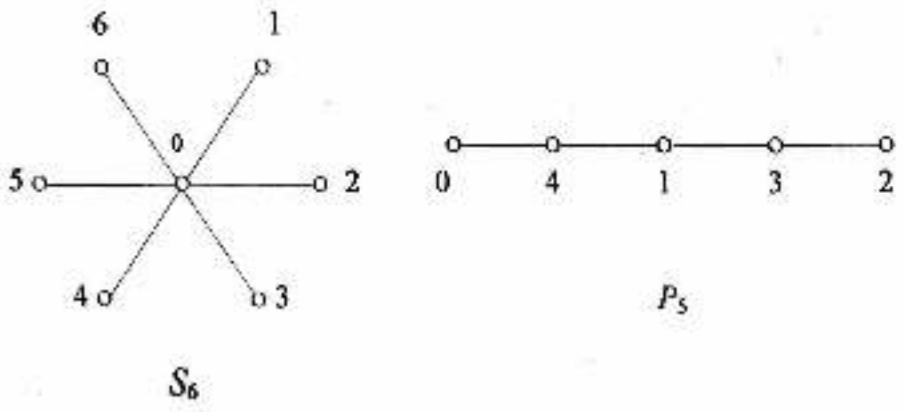

Fig. 2. Graceful labelings of S_6 and P_5.

Harmonious Labeling

We shall prove here that there exist harmonious graphs with arbitrarily large complete subgraphs. This result clearly implies that every graph is a subgraph of some harmonious graph.

The **Fibonacci numbers** F_n are defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for each $n \geq 3$. Let us define the **felicitous numbers** f_n by $f_1 = 0, f_2 = 1$, and $f_n = f_{n-1} + f_{n-2} + 1$, for each $n \geq 3$. One can easily show by mathematical induction that $f_n = F_{n+1} - 1$ for $n \geq 1$.

Consider the first $n \geq 2$ felicitous numbers f_i and let $\Sigma_n = \{ f_i + f_j : i \neq j \text{ and } i, j \leq n \}$. It follows from the definition of felicitous numbers that for each $k \geq 2$, $f_{k+1} > \sigma$ for every $\sigma \in \Sigma_k$. It follows, by mathematical induction, that Σ_n contains $\binom{n}{2}$ distinct elements. Now, 1 and $f_{n+1} - 1$ are the minimum and maximum elements, respectively, of Σ_n. Consequently, $f_{n+1} - 1 \geq \binom{n}{2}$. If we set $\binom{n}{r} = 0$, when $r > n$, then the inequality $f_{n+1} - 1 \geq \binom{n}{2}$ holds for $n \geq 1$. It follows also that $F_{n+2} - \binom{n}{2} - 2 \geq 0$ for $n \geq 1$.

Let S_m be the star with $m \geq 0$ edges. Denote by $K_n \circ S_m$ the graph obtained by identifying one vertex of K_n with the central vertex of S_m. (S_m has a unique central vertex except when $m = 1$ in which case we take anyone of its two central vertices.)

Theorem 1. Let $n \geq 1$, $m = F_{n+2} - \binom{n}{2} - 2$. Then $K_n \circ S_m$ is harmonious.

Proof. The theorem is trivially true when $n = 1$. Assume that $n \geq 2$. Label the vertices of K_n with the felicitous numbers $f_1, f_2, \ldots, f_n$ with the label $f_1 = 0$ at the vertex which is identified with the central vertex of S_m. The number of elements of the set $\{ 1, 2, \ldots, f_{n+1} - 1 \}$ which are not in Σ_n is $f_{n+1} - 1 - \binom{n}{2} = F_{n-2} - \binom{n}{2} - 2 = m \geq 0$. Use these m elements to label the remaining m vertices of S_m. $\square$

Since every graph with n vertices is a subgraph of K_n, we immediately get the following two corollaries.

Corollary 1.1. Every graph is a subgraph of a harmonious graph.

Corollary 1.2. Every graph is a subgraph of a felicitous graph.

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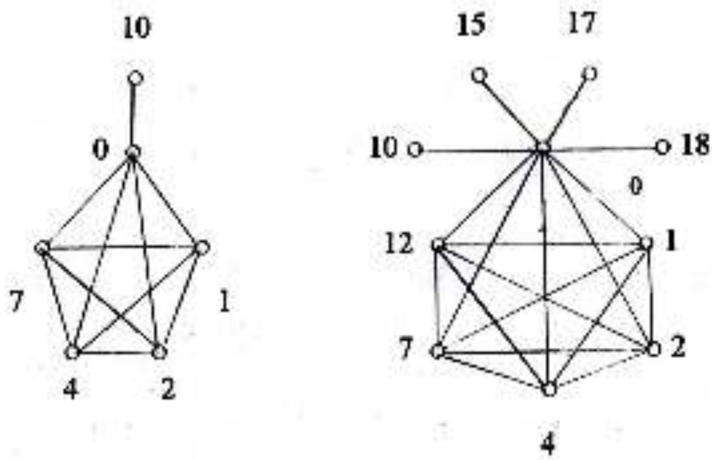
Figure 3 below illustrates the theorem for $n = 5, 6$.



$K_5 \circ S_1$ $K_6 \circ S_4$.

Fig. 3. Harmonious labelings of $K_5 \circ S_1$ and $K_6 \circ S_4$.

Graceful Labeling

Similar to the case of harmonious graphs, we shall prove here that there exist graceful graphs with arbitrarily large complete subgraph.

Let us define the numbers g_n by $g_1 = 0$, and $g_n = 2g_{n-1} + 1$ for $n \geq 2$. It is easy to see that for any positive integer n, the set $\Delta_n = \{|g_i - g_j| : i \neq j, 1 \leq i, j \leq n\}$ consists of $\binom{n}{2}$ distinct elements. From the definition of g_n, it is also easily seen that $g_n = 2^{n-1} - 1$. Since the largest element of Δ_n is g_n, it follows that $g_n \geq \binom{n}{2}$.

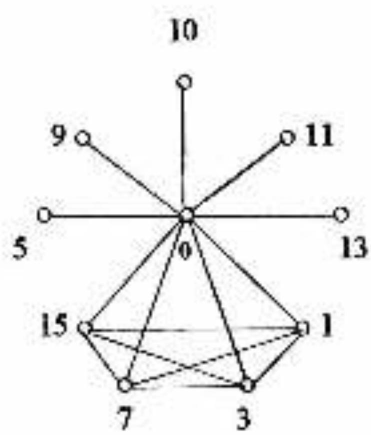
Theorem 2. Let $n \geq 1$ and $m = 2^{n-1} - \binom{n}{2} - 1$. Then the graph $K_n \circ S_m$ is graceful.

Proof. Observe that $m = 2^{n-1} - \binom{n}{2} - 1 = g_n - \binom{n}{2} \geq 0$. Label the vertices of K_n using $g_1, g_2, \ldots, g_n$. Then all the induced edge labels $|g_i - g_j|$, $i = j$ are distinct and they are $\binom{n}{2}$ in number, the total number of edges in K_n . Identify the center of S_m with the vertex of K_n which is labeled by 0.

Use the numbers is the set $\{0, 1, 2, \ldots, \binom{n}{2} + m\}$ to label the other m vertices of S_m. We see then that the induced edge labels are $1, 2, \ldots, \binom{n}{2} + m$, where $\binom{n}{2} + m$ is the number of edges of $K_n \circ S_m$. Therefore, $K_n \circ S_m$ is graceful. □

Corollary 2.1. Every graph is a subgraph of a graceful graph.

Figure 4 below illustrates the theorem for $n = 5$.



$K_5 \circ S_5$

Fig. 4. A graceful labeling of $K_5 \circ S_5$

Remarks

The harmonious graphs and graceful graphs constructed here which have arbitrarily large complete subgraphs contain so many vertices. For example, in constructing a graceful graph which contains K_{10}, we need a star S_{466} and so the graceful $K_{10} \circ S_{466}$ is of order 476. Thus, every graph of order 10 is a subgraph of some graceful graph of order 476. What is the smallest integer k such that every graph of order 10 is a subgraph of some graceful graph of order at most k? Is k equal to 476 or less than 476? The reader is challenged to find more economical constructions of graceful graphs and harmonious graphs containing a complete subgraph of a given order.

References

[1] Gervacio, S. V., *Research Topics in Graph Theory and Combinatorics*, a research project funded by the National Research Council of the Philippines, 1990.

[2] Graham, R. L. and Sloane, N. J. A., *On additive bases and harmonious graphs*, **SIAM J. Alg. Discrete Methods 1** (1980) 382-404.

[3] Lee, S. M., Schmeichel, S. and Shee, S. C., *On felicitous graphs*, **Discrete Math 93** (1991) 201–209