On Some Summation Identities

Roberto B. Corcino
Cristina B. Corcino

Abstract

In this paper we discuss two of these interesting techniques: *interchanging the index of the summations and changing "product of sums" to "sum of products"*. Moreover, we establish a generalization of some well-known summation formulas which consequently give interesting and useful identities.

Keywords: summations, identities, Stirling number, lexicographic, combination

1 Introduction

Several well-known summation identities have already appeared in the literature. Almost all books in combinatorics discussed topics related to summation identities. For instance, the book of Comtet [2], Riordan [5], and Chen, C.C and Koh, K-M [1] made a comprehensive discussion on this topic. However, the book of Comtet [2] has the most complete collection of classical

✍ **ROBERTO B. CORCINO** and **CRISTINA CORCINO** are faculty members of the Department of Mathematics, Mindanao State University, Marawi City. They both hold a Ph.D. degrees in Mathematics from the University of the Philippines, Diliman, Quezon City.

summation identities. For many years, these identities serve as techniques in manipulating expressions containing summation.

2 Some Techniques in Manipulating Summations

Interchanging the Index of the Summations

In deriving identities on some special numbers, such as Stirling numbers, Bell numbers, Fibonacci numbers, and etc., we sometimes make rigorous algebraic manipulations on expression containing multiple summation. Interchanging the index of the summations is one way of simplifying the multiple summation. The following lemma will help us interchange the index of two summations.

Lemma 2.1 $\sum_{i=0}^{n}\sum_{j=0}^{i} F(i,j) = \sum_{j=0}^{n}\sum_{i=j}^{n} F(i,j)$

Proof: The terms at the left-hand side of the equation can be arranged as follows:

i/j	0	1	2	$\cdots$	$n-1$	n
0	$F(0,0)$					
1	$F(1,0)$	$F(1,1)$				
2	$F(2,0)$	$F(2,1)$	$F(2,2)$			
3	$F(3,0)$	$F(3,1)$	$F(3,2)$	$F(3,3)$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
n	$F(n,0)$	$F(n,1)$	$F(n,2)$	$\cdots$	$F(n,n-1)$	$F(n,n)$

$$\sum_{i=0}^{n} F(i,0)+\sum_{i=1}^{n} F(i,1)+\sum_{i=2}^{n} F(i,2)+\cdots+\sum_{i=n-1}^{n} F(i,n-1)+\sum_{i=n}^{n} F(i,n)$$

$$= \sum_{i=0}^{n}\sum_{i=j}^{n} F(i,j)$$

The above single summations are obtained by summing up the terms $F(i,j)$, vertically, which give the preceding double summation. Thus we prove the lemma. □

Note that Lemma 2.1 can be extended further as follows:

$$\sum_{i=k}^{n}\sum_{j=k}^{n} F(i,j) = \sum_{j=k}^{n}\sum_{i=j}^{n} F(i,j) .$$

To illustrate the use of this identity, let us consider the following example.

Example 2.2 This example is taken from the paper of R.B. Corcino [4,p.99] in proving the vertical recurrence relation of (r,β)-Stirling numbers. A portion of the proof gives the following equations

$$\begin{aligned} \sum_{n=k}^{\infty} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\beta,r} t^n &= \sum_{j=k}^{\infty}\sum_{n=j}^{\infty} \left\langle \begin{matrix} j-1 \\ k-1 \end{matrix} \right\rangle_{\beta,r} (\beta k + r)^{n-j} t^n \\ &= \sum_{n=k}^{\infty}\sum_{j=k}^{n} \left\langle \begin{matrix} j-1 \\ k-1 \end{matrix} \right\rangle_{\beta,r} (\beta k + r)^{n-j} t^n \\ \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\beta,r} &= \sum_{j=k}^{n} \left\langle \begin{matrix} j-1 \\ k-1 \end{matrix} \right\rangle_{\beta,r} (\beta k + r)^{n-j} \end{aligned}$$

Note that the second equation is obtained from the first equation by applying the above double summation identity. The third equation is obtained by comparing the coefficients of the term t^n.

Changing "Product of Sums" To "Sum of Products"

The next technique will help us convert the expression in "product of the sums" to expression in "sum of the products". The tool in doing this technique is taken from the book of Comtet [2], which is given by the following transformation

$$\prod_{j=0}^{k}\sum_{n_j\geq 0} F_{n_j} = \sum_{n_0+n_1+\cdots+n_k\geq 0}\prod_{j=0}^{k} F_{n_j}\ .$$

To illustrate the use of this technique, let us consider the following example.

Example 2.3 We recall a portion of the proof of Theorem 4 in the paper of R.B. Corcino [4,p.96] which is given by

$$\begin{aligned}
\sum_{n\geq k}\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\beta,r} t^{n-k} &= \prod_{j=0}^{k}\left[\sum_{c_j\geq 0}(\beta j+r)^{c_j}t^{c_j}\right] \\
&= \sum_{c_0+c_1+\cdots+c_k\geq 0}\left[\prod_{j=0}^{k}(\beta j+r)^{c_j}\right] t^{c_0+c_1+\cdots+c_k} \\
&= \sum_{n\geq k}\left[\sum_{c_0+c_1+\cdots+c_k=n-k}\prod_{j=0}^{k}(\beta j+r)^{c_j}\right] t^{n-k} \\
\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\beta,r} &= \sum_{c_0+c_1+\cdots+c_k=n-k}\prod_{j=0}^{k}(\beta j+r)^{c_j}\ .
\end{aligned}$$

Note that the second equation is obtained from the first equation by applying the above transformation. Furthermore, the last equation is obtained by comparing the coefficients of the term t^{n-k}.

3 A Generalization of Some Well-Known Summation Identities

It is known that

$$\begin{aligned}
1+2+3+\cdots+n &= \frac{n(n+1)}{2} \\
1(2)+2(3)+3(4)+\cdots+n(n+1) &= \frac{n(n+1)(n+2)}{3}\ .
\end{aligned}$$

The following theorem is a generalization of these summation identities.

Theorem 3.1 $\sum_{i=1}^{n}\prod_{j=0}^{k}(i+j)=\frac{1}{k+2}\prod_{j=0}^{k+1}(n+j).$

Proof: We will prove the theorem by double induction, that is, induction on n and k. First, we fix k and do induction on n. For $n=1$, we have

$$\begin{aligned}\sum_{i=1}^{m}\prod_{j=0}^{k}(i+j) &= \prod_{j=0}^{k}(1+j)\\ &= \frac{\prod_{j=0}^{k+1}(1+j)}{k+2}\ .\end{aligned}$$

Assume that for $m>1$, we have

$$\sum_{i=1}^{m}\prod_{j=0}^{k}(i+j)=\frac{\prod_{j=0}^{k+1}(m+j)}{k+2}\ .$$

Then

$$\begin{aligned}\sum_{i=1}^{m+1}\prod_{j=0}^{k}(i+j) &= \sum_{i=1}^{m}\prod_{j=0}^{k}(i+j)+\prod_{j=0}^{k}(m+1+j)\\ &= \frac{\prod_{j=0}^{k+1}(m+j)}{k+2}+\prod_{j=0}^{k}(m+1+j)\\ &= \frac{m\prod_{j=1}^{k+1}(m+j)}{k+2}+\prod_{j=0}^{k}(m+1+j)\end{aligned}$$

$$= \frac{m\prod_{j=0}^{k}(m+1+j)}{k+2} + \prod_{j=0}^{k}(m+1+j)$$

$$= \frac{(m+k+2)\prod_{j=0}^{k}(m+1+j)}{k+2}$$

$$= \frac{\prod_{j=0}^{k+1}(m+1+j)}{k+2} .$$

Now, we fix n and do induction on k. For $k = 0$, we know that

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1) .$$

Assume that for some $k > 0$,

$$\sum_{i=1}^{n}\prod_{j=0}^{k}(i+j) = \frac{1}{k+2}\prod_{j=0}^{k+1}(n+j)$$

Then

$$\begin{aligned}\sum_{i=1}^{n}\prod_{j=0}^{k+1}(i+j) &= \sum_{i=1}^{n} i\prod_{j=0}^{k}(i+j) + (k+1)\sum_{i=1}^{n}\prod_{j=0}^{k}(i+j) \\ &= \sum_{i=1}^{n}\sum_{s=1}^{i}\prod_{j=1}^{k}(i+1) + (k+1)\cdot\frac{1}{k+2}\prod_{j=0}^{k+1}(n+j) \cdot\end{aligned}$$

Using Lemma 2.1, we obtain

$$\begin{aligned}
\sum_{i=1}^{n}\prod_{j=0}^{k+1}(i+j) &= \sum_{s=1}^{n}\sum_{i=s}^{n}\prod_{j=1}^{k}(i+j)+\frac{k+1}{k+2}\prod_{j=0}^{k+1}(n+j) \\
&= \sum_{s=1}^{n}\left[\sum_{i=1}^{n}\prod_{j=1}^{k}(i+j)-\sum_{i=1}^{s-1}\prod_{j=1}^{k}(i+j)\right]+\frac{k+1}{k+2}\prod_{j=0}^{k+1}(n+j) \\
&= \sum_{s=1}^{n}\frac{1}{k+2}\prod_{j=0}^{k+1}(n+j)-\sum_{s=1}^{n}\frac{1}{k+2}\prod_{j=0}^{k+1}(s-1+j) \\
&\quad +\frac{k+1}{k+2}\prod_{j=0}^{k+1}(n+j) \\
&= \frac{n}{k+2}\prod_{j=0}^{k+1}(n+j)-\frac{1}{k+2}\sum_{s=1}^{n-1}\prod_{j=0}^{k+1}(s+j)-\frac{1}{k+2}\prod_{j=0}^{k+1}(n+j) \\
&\quad +\prod_{j=0}^{k+1}(n+j)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{k+3}{k+2}\sum_{i=1}^{n}\prod_{j=0}^{k+1}(i+j) &= \frac{(n+k+2)\prod_{j=0}^{k+1}(n+j)}{k+2} \\
\sum_{i=1}^{n}\prod_{j=0}^{k+1}(i+j) &= \frac{1}{k+3}\prod_{j=0}^{k+2}(n+j)\,.
\end{aligned}$$

This completes the proof of the theorem. □

Example 3.2 Using Theorem 3.1 with $k=4$ and $n=4$, we have

$$\begin{aligned}
1(2)(3)(4)(5)+2(3)(4)(5)(6)+3(4)(5)(6)(7)+4(5)(6)(7)(8) &= \frac{4(5)(6)(7)(8)(9)}{6} \\
&= 10080
\end{aligned}$$

As a direct consequence of Theorem 3.1, we have the following corollary.

Corollary 3.3 $\displaystyle\sum_{i=1}^{n} i \prod_{j=0}^{k-1}(i+j) = \frac{(k+1)n+1}{(k+1)(k+2)} \prod_{j=0}^{k}(n+j).$

Proof: From Theorem 3.1,

$$\sum_{i=1}^{n} i \prod_{j=0}^{k-1}(i+j) + \sum_{i=1}^{k} k \prod_{j=0}^{k-1}(i+j) = \frac{1}{k+2} \prod_{j=0}^{k}(n+j) .$$

Using Theorem 3.1 again, we have

$$\begin{aligned} \sum_{i=1}^{n} i \prod_{j=0}^{k-1}(i+j) &= \frac{1}{k+2} \prod_{j=0}^{k+1}(n+j) - \frac{k}{k+1} \prod_{j=0}^{k}(n+j) \\ &= \frac{1}{k+2} \prod_{j=0}^{k}(n+j) . \qquad \square \end{aligned}$$

Remark 3.4 *When $k = 1$, Corollary 3.3 will reduce to*

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} ,$$

which is a known identity. Furthermore, when $k = 2$, we have

$$\sum_{i=1}^{n} (i^3 + i^2) = \frac{n(n+1)(n+2)(3n+1)}{12} ,$$

which implies that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 .$$

Thus $\displaystyle\sum_{i=1}^{n} i^m$, $m = 4, 5, \ldots$ can also be obtained using Corollary 3.3.

The next consequence of Theorem 3.1 is a simplification of cascaded summations, which is given in the following corollary.

Corollary 3.5 $\displaystyle\sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} \cdots \sum_{i_{m-1}=i_{m-2}}^{n} \sum_{i_m=i_{m-1}}^{n} (n - i_m + 1) = \binom{n+m+1}{m+1}.$

Proof: By repeated application of Theorem 3.1, we have

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\cdots\sum_{i_{m-1}=i_{m-2}}^{n}\sum_{i_m=i_{m-1}}^{n}(n-i_m+1)$$

$$= \sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\cdots\sum_{i_{m-1}=i_{m-2}}^{n}\frac{1}{2}(n-i_{m-1}+1)(n-i_{m-1}+2)$$

$$= \sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\cdots\sum_{i_{m-3}=i_{m-4}}^{n}\sum_{i_{m-2}=i_{m-3}}^{n}\frac{1}{2}\cdot\frac{1}{3}(n-i_{m-2}+1)(n-i_{m-2}+2)(n-i_{m-2}+3)$$

$$= \sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\cdots \sum_{i_{m-3}=i_{m-4}}^{n}\sum_{i_{m-2}=i_{m-3}}^{n}\frac{(n-i_{m-3}+1)(n-i_{m-3}+2)(n-i_{m-3}+3)(n-i_{m-3}+4)}{4!}$$

$$\vdots$$

$$= \sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\frac{(n-i_2+1)(n-i_2+2)\cdots(n-i_2+m-2)(n-i_2+m-1)}{(m-1)!}$$

$$= \sum_{i_1=0}^{n}\frac{(n-i_1+1)(n-i_1+2)\cdots(n-i_1+m-2)(n-i_1+m-1)(n-i_1+m)}{m!}$$

$$= \frac{(n+1)(n+2)\cdots(n+m-2)(n+m-1)(n+m)(n+m+1)}{(m+1)!}$$

$$= \binom{n+m+1}{m+1}. \quad \square$$

Remark 3.6 *The second to the last equation in the proof yields*

$$\sum_{i_2=i_1}^{n}\cdots\sum_{i_{m-1}=i_{m-2}}^{n}\sum_{i_m=i_{m-1}}^{n}(n-i_m+1) = \binom{n-i_1+m}{m}.$$

This identity plays a very important role in proving the main result in [3], which determines the numerical position of a string in a lexicographic ordering where the strings are the r-combinations of the elements of the set $\{1, 2, \ldots, n\}$.

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