Semi-Open Sets and Semi-Closure of a Set

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Abstract

This paper revisits the concept of semi-open set in a topological space defined by N. Levine in 1963 [2]. We also define new concepts such as semi-closed set, and semi-closure point of a set and give some results concerning these concepts.

Keywords: topology, open, semi-open, interior, nowhere dense, semi-closure

1 Introduction

Generally, the family of all semi-open sets in a topological space X, though contains all the open sets, does not form a topology on the set X. In this paper we take another look at semi-open sets and define new concepts such as semi-closed set and semi-closure of a set in a topological space. We shall see that a semi-closure of a set is generally smaller than the closure of a set and that it differs from the latter in some sense.

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Throughout this paper, X, Y, and Z are topological spaces.

2 Definitions and Results

Definition 2.1 A subset O of X is *semi-open* if $O \subseteq cl[int(O)]$ (closure of the interior of O). Equivalently, O is semi-open if there exists an open set G in X such that $G \subseteq O \subseteq cl(G)$.

Remark 2.2 *Every open set is semi-open.*

Remark 2.3 *In a discrete space X, a subset A of X is open if and only if it is semi-open.*

Note that every subset of a discrete space X is open. Thus every semi-open set A in X is open. Therefore by Remark 2.2, the assertion of Remark 2.3 is true.

Theorem 2.4 *Let H be a non-empty nowhere dense set in X, that is, $int(cl(H)) = \varnothing$. Then H^c is semi-open but H is not.*

Proof: To show this, suppose H is semi-open. Then there exists an open set O such that $O \subseteq H \subseteq cl(O)$. Since H is non-empty, O is non-empty. Thus $int(H)$ (which contains O) is non-empty. It follows that $int(cl(H)) = \varnothing$ contrary to our assumption that H is nowhere dense. Therefore, H is not semi-open.

Next, let G be the complement of $cl(H)$. Since $cl(H)$ is a closed set containing H, it follows that G is an open set and $G \subseteq H^c$. Also, $cl(G) = X$ (otherwise, $int(cl(H)) \neq \varnothing$). Thus we have

$$G \subseteq H^c \subseteq cl(G) .$$

Therefore H^c is a semi-open set. □

Corollary 2.5 *If O is an open subset of X that is not closed, then $cl(O) \setminus O$ is not semi-open in X.*

Proof: By Theorem 2.4, since $cl(O) \setminus O \neq \varnothing$, it remains to show that $cl(O) \setminus O$ is nowhere dense. So, suppose that $x \in int[cl(cl(O) \setminus O)]$. Then there exists an open set G_x containing x such that $G_x \subseteq cl(cl(O) \setminus O)$. Since

$$cl(cl(O) \setminus O) \subseteq cl(O^c) = O^c ,$$

it follows that $G_x \cap O = \varnothing$. Thus, $x \notin cl(O)$. But $x \in int[cl(cl(O) \setminus O)]$ implies that $x \in cl(O)$ because

$$int[cl(cl(O) \setminus O)] \subseteq cl(cl(O) \setminus O) \subseteq cl(cl(O)) = cl(O) .$$

Therefore, we obtain a contradiction.
Therefore $int[cl(cl(O) \setminus O)] = \varnothing$, i.e., $cl(O) \setminus O$ is nowhere dense in X. □

Corollary 2.6 *If S is a semi-open subset of X that is not closed, then $cl(S) \setminus S$ is not semi-open in X.*

Proof: We claim that the difference $cl(S) \setminus S$ is a non-empty nowhere dense set in X. Since S is not closed, $cl(S) \setminus S \neq \varnothing$. Now, since S is semi-open, there exists an open set O such that $O \subseteq S \subseteq cl(O)$. Clearly, $cl(S) = cl(O)$. Thus $cl(S) \setminus S \subseteq cl(O) \setminus O$. Our proof of Corollary 2.5 shows that $cl(O) \setminus O$ is nowhere dense in X. Therefore the inclusion

$$cl(S) \setminus S \subseteq cl(O) \setminus O$$

implies that $cl(S) \setminus S$ is also nowhere dense in X. The desired result now follows from Theorem 2.4. □

Theorem 2.7 (Levine) *Let $\{O_\alpha : \alpha \in I\}$ be a collection of semi-open sets in X. Then $\cup\{O_\alpha : \alpha \in I\}$ is a semi-open set in X.*

Remark 2.8 *The intersection of two semi-open sets need not be semi-open.*

To see this, let X be the real space with the standard topology, $A = (0, 2]$ and $B = [2, 3)$. Clearly, A and B are semi-open sets in X. However, $A \cap B = \{2\}$ is not semi-open in X because

$$\{2\} \not\subseteq cl(int(\{2\})) = cl(\varnothing) = \varnothing .$$

Definition 2.9 A subset F of X is *semi-closed* if its complement F^c is semi-open in X.

Remark 2.10 *Every closed subset of a topological space is semi-closed.*

Note that Remark 2.10 follows from Remark 2.2 and Definition 2.9.

Theorem 2.11 *Let $\{F_\alpha : \alpha \in I\}$ be a collection of semi-closed sets in X. Then $\cap\{F_\alpha : \alpha \in I\}$ is a semi-closed set in X.*

Proof: By applying De Morgan's law and using Theorem 2.7, we get the desired result. □

Next, we define the concept of semi-closure point of a set.

Definition 2.12 Let A be a subset of X. A point $p \in X$ is a *semi-closure point* of A if for every semi-open set G in X, $p \in G$ implies that $G \cap A \neq \varnothing$. We denote by $scl(A)$ the set of all semi-closure points of A.

Theorem 2.13 *Let* $A \subseteq X$. *Then*

(a) $A \subseteq scl(A)$;

(b) $scl(A) \subseteq cl(A)$;

(c) A *is semi-closed if and only if* $A = scl(A)$.

Proof: (a) Let $p \in A$ and G a semi-open set with $p \in G$. Then $p \in A \cap G$ and so, $A \cap G \neq \varnothing$. This shows that $p \in scl(A)$. Therefore, $A \subseteq scl(A)$.

(b) Let $x \in scl(A)$ and let G be an open set in X with $x \in G$. By Remark 2.2, G is a semi-open set containing x. Since $x \in scl(A)$, $G \cap A \neq \varnothing$ by Definition 2.12. This shows that $x \in cl(A)$. Accordingly, $scl(A) \subseteq cl(A)$.
(c)(:) Suppose A is semi-closed. Let $x \notin A$, i.e., $x \in A^c$. Set $G = A^c$. Then G is semi-open by Definition 2.9 and $x \in G$. Since $A \cap G = \varnothing$, it follows that $x \in scl(A)$. Therefore, $scl(A) \subseteq A$. Combining this with (a), we have $A = scl(A)$.

($\Leftarrow$) Suppose $A = scl(A)$ and let $p \notin A^c$. Since $p \notin scl(A)$, there exists a semi-open set O_p containing p such that $O_p \cap A = \varnothing$, i.e., $O_p \subseteq A^c$. It follows that

$$A^c = \cup\{O_p : p \in A^c\} .$$

Therefore, A^c is semi-open by Theorem 2.7. Therefore, A is semi-closed by Definition 2.9.

The proof of the theorem is complete. □

Remark 2.14 *There exists a set* A *with* $scl(A)$ *properly contained in* $cl(A)$.

Let $X = \mathbb{R}$ with the standard topology. Consider $A = (0, 2)$. By definition of closure of a set, $cl(A) = [0, 2]$. Now, since $(-\infty, 0]$ and $[2, +\infty)$ are semi-open sets, $(-\infty, 0] \cup [2, +\infty) = A^c$ is semi-open. It follows that A is semi-closed. Thus, by Theorem 2.13(c), $scl(A) = (0, 2)$. This justifies the remark.

Theorem 2.15 *Let $A \subseteq X$. Then $scl(A)$ is the smallest semi-closed set containing A, i.e., $A = \cap\{F : F$ is semi-closed and $A \subseteq F\}$. Moreover, for $B, C \subseteq X$, we have the following:*

(a) $B \subseteq C$ *implies that* $scl(B) \subseteq scl(C)$

(b) $scl(scl(B)) = scl(B)$

(c) $scl(B) \cup scl(C) \subseteq scl(B \cup C)$

(d) If A is closed then $scl(A) = A$. In particular, $scl(\varnothing) = \varnothing$.

Proof: Let $T = \cap\{F : F$ is semi-closed and $A \subseteq F\}$. It is easy to see that $A \subseteq T$. Also, by Theorem 2.11, T is a semi-closed set in X. Hence, T^c is semi-open in X. Thus, if $p \notin T$, then $p \notin scl(A)$ because

$$T^c \cap A \subseteq T^c \subseteq T = \varnothing$$

This means that $scl(A) \subseteq T$.

On the other hand, if $z \notin scl(A)$, then there exists a semi-open set G_z containing z such that $G_z \cap A = \varnothing$, i.e., $A \subseteq G_z^c$, where G_z^c is a semi-closed set. Since $z \notin G_z^c$, $z \notin T$ by definition of the set T. It follows that $T \subseteq scl(A)$.

The above inclusions imply the desired equality.

Now, suppose that A, B and C are subsets of X. If $B \subseteq C$, then the set

$$\{F : F \text{ is semi-closed and } C \subseteq F\}$$

is contained in

$$\{K : K \text{ is semi-closed and } B \subseteq K\} .$$

Thus, using the result of the first part of the theorem, $scl(B) \subseteq scl(C)$. This proves (a). Since $scl(A)$ is semi-closed, (b) follows from Theorem 2.13(c). The statements in (d) follow from Remark 2.10 and Theorem 2.13(c). Finally, because $B \subseteq B \cup C$ and $C \subseteq B \cup C$, (c) will follow from (a). □

Remark 2.16 *There exist sets B and C such that $scl(B) \cup scl(C)$ is properly contained in $scl(B \cup C)$.*

Again, consider the real space $X = \mathbb{R}$ with the standard topology. Let $B = (0, 2]^c$ and $C = [2, 3)^c$. Then B and C are semi-closed sets (see Remark 2.8). Hence, by Theorem 2.13(c),

$$scl(B) \cup scl(C) = B \cup C = \mathbb{R} \setminus \{2\} .$$

Since $R \setminus \{2\}$ is not semi-closed, the smallest semi-closed set containing it is $\mathbb{R}$. Therefore, by Theorem 2.15, $scl(B \cup C) = \mathbb{R}$. This proves the remark.

Remark 2.16 simply tells us that the operation "semi-closure" on a set is not distributive over unions of sets. This is one difference between the two closure operations.

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The Fundamental Theorem and the Cauchy Extension for Integrals in Local Systems

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Abstract

A version of the Fundamental Theorem of Calculus for Wang and Ding's integral is formulated in this paper. At the same time, it also states and proves the Cauchy Extension Theorem for the said integral as defined using Thomson's local system.

Keywords: local system, choice, S-cover, S-integral, Henstock's lemma

1 Preliminary Concepts and Results

We first give the concepts and preliminary results that we need.

Definition 1.1 Let $\mathbb{R}$ be the real line and $2^{\mathbb{R}}$ the collection of all subsets of $\mathbb{R}$. Suppose for every $x \in \mathbb{R}$, there corresponds a nonempty $S(x) \subset 2^{\mathbb{R}}$ such that

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1. $\{x\} \notin S(x)$;

2. $x \in \sigma$ whenever $\sigma \in S(x)$;

3. $\sigma_1 \in S(x)$ and $\sigma_1 \subset \sigma_2$ implies that $\sigma_2 \in S(x)$;

4. $\sigma \in S(x)$ and $\delta > 0$ implies that $(x - \delta, x + \delta) \cap \sigma \in S(x)$.

Then $S = \{S(x) : x \in \mathbb{R}\}$ is called a **local system**. S is said to be **bilateral** if every $\sigma \in S(x)$ contains points on both sides of x. It is **filtering** if for every $x \in \mathbb{R}$, $\sigma_1 \cap \sigma_2 \in S(x)$ whenever $\sigma_1, \sigma_2 \in S(x)$. Further, it is said to have the **intersection condition** if for every collection of sets $\{\sigma(x) : x \in \mathbb{R}\}$, called a **choice from** S, where $\sigma(x) \cap S(x)$, there is a positive function δ such that if $0 < y - x < \min\{\delta(x), \delta(y)\}$, then $\sigma(x) \cap \sigma(y) \cap [x, y] \neq \varnothing$.

Definition 1.2 A family C of finite closed intervals is called an S-cover of $\mathbb{R}$ if for every $x \in \mathbb{R}$, the set

$$\sigma(x) = \{y : \ y = x, \text{ or } y > x \text{ and } [x, y] \in C, \text{ or } y < x \text{ and } [y, x] \in C\}$$

belongs to $S(x)$.

The following results are proved in [1].

Lemma 1.3 *A family C of finite closed intervals is an S-cover of $\mathbb{R}$ if and only if there exists a choice $\{\sigma(x) : x \in \mathbb{R}\}$ of S such that*

(a) *if $y \in \sigma(x)$ and $y > x$, then $[x, y] \in C$ and*

(b) *if $y \in \sigma(x)$ and $y < x$, then $[y, x] \in C$*

Lemma 1.4 *Let* $\{\sigma(x) : x \in \mathbb{R}\}$ *be choice from* S *and*

$$C = \{[u, v] \subset \mathbb{R} : u \in \tau(v) \text{ or } v \in \tau(u)\}.$$

Then C *is an* S*-cover of* $\mathbb{R}$.

In view of Lemma 1.3, it follows that every S-cover C of $\mathbb{R}$ has a corresponding choice from S. Conversely, every choice from S can be used to define an S-cover for $\mathbb{R}$. These facts will be frequently used in the next section.

Definition 1.5 Let $S = \{S(x) : x \in \mathbb{R}\}$ be a local system which is bilateral and has the intersection condition and C an S-cover for $\mathbb{R}$. A tagged division $D = \{([u, v]; \xi)\}$ is called a C**-partition** of an interval $[a, b]$ if each $[u, v]$ belongs to C. The associated or tag point is $\xi = u$ if $v \in \sigma(u)$ or $\xi = v$ if $u \in \sigma(v)$ or either one when both occur, where $\{\sigma(x) : x \in \mathbb{R}\}$ is the choice from S corresponding to C.

The following result guarantees the existence of a C-partition of an interval for every given S-cover C of R. See [4] for its proof.

Theorem 1.6 (Thomson's Lemma) *Let* $S = \{S(x) : x \in \mathbb{R}\}$ *be a local system which is bilateral and has the intersection condition. If* C *is an* S*-cover of* $\mathbb{R}$*, then there is a* C*-partition of any interval* $[a, b]$.

Henceforth, $S = \{S(x) : x \in \mathbb{R}\}$ is a fixed local system that is bilateral, filtering and satisfies the intersection condition.

Definition 1.7 The real number L is the *S-limit* of a function $F : [a,b] \to \mathbb{R}$ as $x \to \xi$, if for every $\epsilon > 0$, there exists a $\sigma(\xi) \in S(\xi)$ such that $|F(x) - L| < \epsilon$ whenever $x \in \sigma(\xi) \cap [a,b]$ and $x \neq \xi$. In this case, we write

$$S\text{-}\lim_{x \to \xi} F(x) = L \ .$$

Theorem 1.8 [1] If $\lim_{x \to \xi} F(x) = L$, then $S\text{-}\lim_{x \to \xi} F(x) = L$.

Definition 1.9 A function $F : [a,b] \to \mathbb{R}$ is *S-differentiable* at $\xi \in [a,b]$ if

$$S\text{-}\lim_{x \to \xi} \frac{F(x) - F(\xi)}{x - \xi} \quad \text{exists.}$$

We denote this limit by $SDF(\xi)$ and call it the *S-derivative* of F at ξ. F is *S-differentiable* on $[a,b]$ if it is S-differentiable at every point in $[a,b]$.

Corollary 1.10 *If $F : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$, then it is S-differentiable there. Moreover, $F'(x) = SDF(x)$ for every $x \in [a,b]$.*

Proof: This follows from Definition 1.9 and Theorem 1.8. □

Definition 1.11 A function $F : [a,b] \to \mathbb{R}$ is *S-continuous* at $\xi \in [a,b]$ if for every $\epsilon > 0$, there exists $\sigma(\xi) \in S(\xi)$ such that $|F(x) - F(\xi)| < \epsilon$ whenever $x \in \sigma(\xi) \cap [a,b]$. F is *S-continuous* on $[a,b]$ if it is continuous at every point in $[a,b]$.

Theorem 1.12 *If $F : [a,b] \to \mathbb{R}$ is S-differentiable on $[a,b]$, then it is S-continuous there.*

Proof: Let $\xi \in [a,b]$ and let $\epsilon > 0$. By assumption, corresponding to $\epsilon/(b-a)$, there exists a $\sigma(\xi) \in S(\xi)$ such that

$$|F(x) - F(\xi) - SDF(\xi)(x-\xi)| < \epsilon$$

whenever $x \in \sigma(\xi) \cap [a,b]$ and $x \neq \xi$. Set

$$\tau(\xi) = \sigma(\xi) \cap (\xi - \delta, \xi + \delta) ,$$

where

$$\delta = \epsilon/(1 + |SDF(\xi)|) .$$

By Property (4) in Definition 1.1, it follows that $\tau(\xi) \in S(\xi)$. Now, if $x \in \tau(\xi) \cap [a,b]$, then we have

$$\begin{aligned} |F(x) - F(\xi)| &\leq |F(x) - F(\xi) - SDF(\xi)(x-\xi)| + |SDF(\xi)||x-\xi| \\ &< 2\epsilon . \end{aligned}$$

This shows that F is S-continuous at ξ. Therefore F is S-continuous on $[a,b]$. □

Lemma 1.13 *If $\xi \in \mathbb{R}$ and $\delta > 0$, then $(\xi - \delta, \xi + \delta) \in S(\xi)$.*

Proof: By Property (3) of Definition 1.1, the real line $\mathbb{R}$ is in $S(\xi)$. Thus, by Property (iv),

$$(\xi - \delta, \xi + \delta) = (\xi - \delta, \xi + \delta) \cap R \in S(\xi) .$$ □

Theorem 1.14 *If $F : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then it is S-continuous there.*

Proof: Let $\xi \in [a,b]$ and let $\epsilon > 0$. By continuity of F at ξ, there exists a $\delta > 0$ such that $|F(x) - F(\xi)| < \epsilon$ whenever $x \in (\xi - \delta, \xi + \delta) \cap [a,b]$. Put $\sigma(\xi) = (\xi - \delta, \xi + \delta)$. Then $\sigma(\xi) \in S(\xi)$ by Lemma 1.13. Further, if $x \in \sigma(\xi) \cap [a,b]$, then $|F(x) - F(\xi)| < \epsilon$. This proves the theorem. □

Definition 1.15 A real-valued function f defined on $[a,b]$ is **S-integrable** to the number A if for every $\epsilon > 0$ there exists an S-cover C of $\mathbb{R}$ such that for any C-partition $D = \{([u,v];\xi)\}$ of $[a,b]$,

$$\left|(D)\sum f(\xi)(v-u) - A\right| < \epsilon.$$

The S-integral A of f, if it exists, is unique. In symbols we write

$$(S)\int_a^b f(t)\,dt = (S)\int_a^b f = A.$$

It is shown in [1] and [5] that the set of all S-integrable real-valued functions defined on $[a,b]$ is a real vector space and that S-integrability over the whole interval implies S-integrability on every subinterval. Further, the following results were also proved.

Theorem 1.16 *If $f(x) = 0$ almost everywhere in $[a,b]$ then f is S-integrable to zero on $[a,b]$.*

Corollary 1.17 *Let f and g be real-valued functions on $[a,b]$. If f is S-integrable on $[a,b]$ and $f = g$ almost everywhere on $[a,b]$, then g is also S-integrable on $[a,b]$. Moreover,*

$$(S)\int_a^b f = (S)\int_a^b g.$$

Theorem 1.18 (Henstock's Lemma) *If $f : [a,b] \to \mathbb{R}$ is S-integrable on $[a,b]$ with S-primitive F defined by*

$$F(x) = (S)\int_a^x SDF(t)\,dt = F(x) - F(a)$$

for every $x \in [a,b]$, then for every $\epsilon > 0$, there exists an S-cover C of $\mathbb{R}$ such that for any C-partition $D = \{([u,v];\xi)\}$ of $[a,b]$,

$$\left|(D)\sum f(\xi)(v-u) - F(v) + F(u)\right| < \epsilon.$$

2 Main Results

The first result says that the S-integral recovers a function (an S-continuous function) from its S-derivative.

Theorem 2.1 (Fundamental Theorem) *Let $F : [a,b] \to \mathbb{R}$ be S-continuous on $[a,b]$. If F is S-differentiable nearly everywhere on $[a,b]$(i.e. except for a countable set), then SDF is S-integrable on $[a,b]$ and*

$$(S)\int_a^x SDF(t)\,dt = F(x) - F(a)$$

for every $x \in [a,b]$.

Proof: Let $E = \{x_n \in [a,b] : SDF(x_n) \text{ does not exist}\}$. Define $G : [a,b] \to \mathbb{R}$ by

$$G(x) = SDF(x)$$

if $x \in [a,b] \setminus E$ and $G(x) = 0$ if $x \in E$. Then $G(x) = SDF(x)$ nearly everywhere. We shall show that G is S-integrable on $[a,b]$.

So, let $\epsilon > 0$. By S-differentiability of F on $[a,b] \setminus E$, there exists $\sigma(x) \in S(x)$ for each $x \in [a,b] \setminus E$ such that if $y \in \sigma(x) \cap [a,b]$ and $y \neq x$, then

$$|F(y) - F(x) - SDF(x)(y-x)| < \epsilon \cdot |y-x|.$$

For $x = x_n$, use S-continuity of F at x to choose $\tau(x) \in S(x)$ so that

$$y \in \tau(x) \cap [a,b] \text{ implies } |F(y) - F(x)| < \epsilon \cdot 2^{-n}.$$

Next, define $v(x) = \sigma(x)$ if $x \in [a,b] \setminus E$, $v(x) = \tau(x)$ if $x \in E$, and $v(x) = \mathbb{R}$ if otherwise. Then $v(x) \in S(x)$ for every real number x and $M = \{v(x) : x \in \mathbb{R}\}$ is a choice from S. Let C be an S-cover of $\mathbb{R}$ corresponding to M and $D = \{([u,v];\xi)\}$ a C-partition of $[a,b]$. Write $D = D_1 \cup D_2$, where $D_1 = \{([u,v];\xi) \in D : \xi \in E\}$ and $D_2 = \{([u,v];\xi) \in D : \xi \in [a,b] \setminus E\}$. Then

$$\begin{aligned}
&|(D)\textstyle\sum G(\xi)(v-u) - F(b) + F(a)| \\
&\quad \leq \; |(D_1)\sum G(\xi)(v-u) - F(b) + F(a)| \\
&\qquad\qquad + |(D_2)\sum G(\xi)(v-u) - F(b) + F(a)| \\
&\quad \leq \; (D_1)\sum |G(\xi)(v-u) - F(v) + F(u)| + (D_2)\sum |F(v) - F(u)| \\
&\quad < \; \epsilon \cdot (b-a) + \sum_{n=1}^{\infty} \epsilon \cdot 2^{-n} \\
&\quad = \; (b-a+1)\epsilon.
\end{aligned}$$

This shows that G is S-integrable on $[a,b]$ and

$$(S)\int_a^b G(t)\,dt = F(b) - F(a).$$

Therefore, by Theorem 1.18, SDF is S-integrable and

$$(S)\int_a^b SDF(t)\,dt = F(b) - F(a).$$

The above argument can be applied to any interval $[a, x]$ $\forall x \in [a, b]$. □

Lemma 2.2 *Suppose that* $(S) \int_a^b f(t)\, dt$ *exists for every* $x \in (a, b)$. *Then for every* $\epsilon > 0$, *there exists an* S*-cover* C *of* $\mathbb{R}$ *such that if* $c \in (a, b)$ *and* $D = \{([u, v]; \xi)\}$ *is a* C*-partition of* $[c, b]$, *then*

$$\left|(D) \sum f(\xi)(v-u) - (S) \int_a^b f\right| < \epsilon.$$

Proof: Let $\{a_n\}$ be a decreasing sequence in (a, b) that converges to a. Set $a_0 = b$ and let $\epsilon > 0$. For each n, there exists an S-cover C_n of $\mathbb{R}$ such that if $D = \{([u, v]; \xi)\}$ is a C_n-partition of $[a_n, a_{n-1}]$, we have

$$\left|(D) \sum f(\xi)(v-u) - (S) \int_u^v f\right| < \epsilon \cdot 2^{-n}.$$

For each n, let $\{\sigma_n(x) : x \in \mathbb{R}\}$ be a choice from S corresponding to C_n. Define

$$\tau(x) = \begin{cases} \sigma_1(x) \cap (a_1 + \frac{3}{4}(a_0 - a_1), a_0 + \frac{1}{4}(a_0 - a_1)) & \text{, if } x = a_0 = b \\ \sigma_n(x) \cap (a_n, a_{n-1}) & \text{, if } a_n < x < a_{n-1} \text{ for some } n \\ \sigma_n(x) \cap (a_{n+1}, a_{n-1}) & \text{, if } x = a_n \text{ for some } n \\ \mathbb{R} & \text{, elsewhere.} \end{cases}$$

Then $\{\tau(x) : x \in \mathbb{R}\}$ is a choice from S. Let

$$C = \{[u, v] \subset \mathbb{R} : u \in \tau(v) \text{ or } v \in \tau(u)\}.$$

Then by Lemma 1.2, C is an S-cover of $\mathbb{R}$. Let $c \in (a, b)$ and $D = \{([u, v]; \xi)\}$ a C-partition of $[c, b]$. Choose an integer m such that

$$a_{m+1} < c < a_m .$$

Note that by definition of $\tau(x)$, a_n is an associated point of D for every $n = 1, 2, \ldots, m$. Also, for every interval-point pair $([u, v]; \xi)$ in D, there exists some $n = 1, 2, \ldots, m+1$ such that $[u,v] \subset [a_n, a_{n-1}]$. For such an n, let D_n denote the set of pairs $([u,v];\xi)$ in D such that $[u,v] \subset [a_n, a_{n-1}]$. Note that D_n is a partial C_n-partition of $[a_n, a_{n-1}]$. Also, observe that

$$D_{m+1} = D \setminus \bigcup_{n=1}^{m} D_n$$

is a partial C_{m+1}-partition of $[a_{m+1}, a_m]$. Thus,

$$\begin{aligned} \left|(D)\sum f(\xi)(v-u) - (S)\int_c^b f\right| &= \left|(D)\sum \left\{f(\xi)(v-u) - (S)\int_u^v f\right\}\right| \\ &\leq \sum_{n=1}^{m+1} \left|(D_n)\sum \left\{f(\xi)(v-u) - (S)\int_u^v f\right\}\right| \\ &< \sum_{n=1}^{\infty} \epsilon \cdot 2^{-n} = \epsilon\ . \end{aligned}$$

This completes the proof of the theorem. □

We now prove the **Cauchy Extension Theorem.**

Theorem 2.3 (Cauchy Extension) *(i) If $f : [a,b] \to \mathbb{R}$ is S-integrable on each interval $[x,b]$ for every $x \in (a,b]$ and $(S)\lim_{x\to a}\int_x^b f$ exists, then f is S-integrable on $[a,b]$ and*

$$(S)\int_a^b f = (S)\lim_{x\to a}\int_x^b f.$$

(ii) If $f : [a,b] \to \mathbb{R}$ is S-integrable on each interval $[a,x]$ for every $x \in [a,b)$ and $(S)\lim_{x\to b}\int_a^x f$ exists, then f is S-integrable on $[a,b]$ and

$$(S)\int_a^b f = (S)\lim_{x\to b}\int_a^x f.$$

Proof: Let

$$L = (S)\lim_{x\to a}\int_x^b f$$

and let $\epsilon > 0$. By Lemma 2.2, there exists an S-cover $C*$ of $\mathbb{R}$ such that if $x \in (a,b)$ and $D = \{([u,v];\xi)\}$ is a $C*$-partition of $[x,b]$, then

$$\left|(D)\sum f(\xi)(v-u) - (S)\int_x^b f\right| < \epsilon.$$

Let $\{\sigma * (x) : x \in \mathbb{R}\}$ be a choice corresponding to $C*$. Since $(S)\lim_{x\to a}\int_x^b f$ exists, there exists $\sigma(a) \in S(a)$ such that whenever $x \in \sigma(a) \cap [a,b]$, we have

$$\left|(S)\int_x^b f - L\right| < \epsilon.$$

Define

$$\tau(x) = \begin{cases} \sigma * (x) \cap (a, x+\epsilon) & , \text{ if } x \in (a,b] \\ \sigma(a) \cap (a - \frac{\epsilon}{1+|f(a)|}, a + \frac{\epsilon}{1+|f(a)|}) & , \text{ if } x = a \\ \mathbb{R} & , \text{ elsewhere} \end{cases}$$

Let $C = \{[u,v] \subset \mathbb{R} : u \in \tau(v) \text{ or } v \in \tau(u)\}$. Then by Lemma 1.2, C is an S-cover of $\mathbb{R}$. Let $D = \{([u,v];\xi)\}$ a C-partition of $[a,b]$. There exists a c such that $([a,c];\xi)$ is in D. Since $a \notin \tau(c)$, it follows that $\xi = a$. Write $D = D_1 \cup D_2$, where $D_2 = \{([a,c];a)\}$ and $D_1 = D \setminus D_2$. Then

$$\begin{aligned} \left|(D)\sum f(\xi)(v-u) - L\right| &= \left|(D_1)\sum f(\xi)(v-u) + f(a)(c-a) - L\right| \\ &\leq \left|(D_1)\sum f(\xi)(v-u) - (S)\int_c^b f\right| \\ &\quad + \left|(S)\int_c^b f - L\right| + |f(a)|(c-a) \\ &< \epsilon + \epsilon + \epsilon \\ &= 3\epsilon. \end{aligned}$$

This proves that f is S-integrable on $[a,b]$. Part (ii) can be proved using a similar argument. □

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