Applications of Cousin's Lemma

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Cousin's Lemma is one of the many equivalent formulations of the Completeness Axiom and the Heine-Borel Theorem. It is a profound idea, but it can be stated in simple terms, and it is a powerful and elegant tool in the proofs of real analysis theorems. This lemma was discovered in 1894 (see [ST; p. 645]), but it has remained obscure until quite recently when it was found to be essential in the study of the Henstock and other Riemann-type integrals. The lemma has a curious habit of rediscovery. Several authors, e.g., see [B1], [B2], [F], [J], [Li], [MR] and [S], have found different versions with similar applications. In this paper we shall give a fairly detailed account of the same applications using Cousin's Lemma. Introducing this lemma in real analysis courses opens the possibility for a unified and simplified approach to the study of analysis.

Assumptions. A closed interval $[a,b]$ is fixed throughout. The letter δ will always denote a positive function defined on $\mathbf{R}$. In defining δ, it is enough to specify $\delta(x)$ when x is in $[a,b]$. For convenience, we shall assume that $\delta(x) = 1$ outside $[a,b]$. When $\delta(x)$ is given, we put the interval

$$\mathbf{O}(x) = (x-\delta(x), x+\delta(x))$$

for each $x \in \mathbf{R}$. The letter $\mathbf{I}$ or the interval $[u,v]$ will always denote a closed subinterval of $[a,b]$. The Lebesgue measure of a set $\mathbf{S}$ is denoted by $|\mathbf{S}|$; in particular the length of an interval $\mathbf{I}$ is denoted by $|\mathbf{I}|$.

1.0 Definition. Let $\delta(x)$ be a positive function defined on $\mathbf{R}$. A **partial division** of $[a,b]$ is a collection $D = \{\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n\}$ of finitely many non-overlapping closed subintervals of $[a,b]$. If $[a,b] = \bigcup_{k=1}^{n} \mathbf{I}_k$, we say that the collection D is a **division** of $[a,b]$. For any closed subinterval $\mathbf{I}$ of $[a,b]$ in D, if there is $\xi \in \mathbf{I}$ such that

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$$\mathbf{I} \subseteq \mathbf{O}(\xi) = (\xi - \delta(\xi), \xi + \delta(\xi)), \tag{1.1}$$

we say that **I** is **δ-fine**. The element ξ is called the **tag** of the subinterval **I**. If all subintervals **I** in D are δ-fine, we say that D is a δ–**fine division** (or **partial division**) of $[a,b]$. Following others, we denote the division D by

$$D = \{([u,v];\xi)\},$$

where $[u,v]$ is a typical tagged interval in D, and ξ, its tag.

The symbol $(D)\sum$ ... means that the sum ranges over all members of the collection D.

2.0 Cousin's Lemma. Let $\delta : \mathbf{R} \to (0,\infty)$ be a positive function on **R**. Then every finite interval $[a,b]$ has a δ–fine division.

Proof. Our proof is based on the Completeness Axiom. Consider the set

$$\mathbf{S} = \{\, x \in [a,b] : \text{there is a } \delta\text{–fine division of } [a,x] \,\}.$$

Since $\delta(a) > 0$, there is a closed subinterval $[a,x] \subseteq \mathbf{O}(a) \cap [a,b]$. Hence the interval $[a,x]$ is δ–fine, and it follows that $[a,x] \subseteq \mathbf{S}$.

Since $\mathbf{S} \neq \varnothing$, and **S** is bounded above by b, it follows from the Completeness Axiom that **S** has a least upper bound σ. Clearly $a < \sigma \leq b$.

Claim: $\sigma \in \mathbf{S}$ and $\sigma = b$. Since $\mathbf{O}(\sigma)$ is open, there is an element x of **S** such that $a < x \leq \sigma$ and $[x,\sigma] \subseteq \mathbf{O}(\sigma)$. Thus the interval $[x,\sigma]$ is δ–fine. Because $x \in \mathbf{S}$, it follows that $\sigma \in \mathbf{S}$.

To prove that $\sigma = b$, suppose, to get a contradiction, that $\sigma < b$. Now let $\sigma < s \leq b$ such that $[\sigma,s] \subseteq \mathbf{O}(\sigma)$. Then $[\sigma,s]$ is δ–fine. Since σ is in S, it follows that $s \in$ S. But this contradicts the definition of σ. Therefore we must have $\sigma = b$. This proves that $b \in \mathbf{S}$ and, hence, by definition of S, there is a δ–fine division of the interval $[a,b]$. □

As illustration of the technique in the use of Cousin's Lemma, we shall prove theorems from elementary real analysis including the Heine-Borel, and the Bolzano-Weierstrass Theorems.

3.0 Theorem. If f is continuous on $[a,b]$, then f is bounded.

Proof. Let $x \in [a,b]$. Since f is continuous at x, there exists a positive number $\delta(x)$ such that $|f(y) - f(x)| < 1$ for all $y \in \mathbf{O}(x) \cap [a,b]$, where $\mathbf{O}(x) = (x-\delta(x), x+\delta(x))$. Hence, if $y \in \mathbf{O}(x) \cap [a,b]$, then

$$f(x) - 1 < f(y) < f(x) + 1. \tag{3.1}$$

By Cousin's Lemma, let $D = \{(\mathbf{I}_k;\xi_k) : 1 \leq k \leq n\}$ be a δ–fine division of $[a,b]$. It follows from (3.1) that if $y \in \mathbf{I}_k$, then $f(\xi_k) - 1 < f(y) < f(\xi_k) + 1$. Thus if $y \in [a,b]$, then, with the maximum and minimum taken over all k, for $1 \leq k \leq n$, we have

$$\min_k f(\xi_k) - 1 \leq f(y) \leq \max_k f(\xi_k) + 1.$$

This shows that f is bounded on $[a,b]$. □

4.0 Heine–Borel Theorem. An open cover **C** of $[a,b]$ has a finite subcover.

Proof. Let **C** be an open cover of $[a,b]$. By hypothesis there is for each x in $[a,b]$ an open set $\mathbf{G}(x)$ in **C** containing x. Hence there is a $\delta(x) > 0$ such that $\mathbf{O}(x) \subseteq \mathbf{G}(x)$. This defines a positive function δ on $[a,b]$.

By Cousin's Lemma, let $D = \{(\mathbf{I}_k,\xi_k) : 1 \leq k \leq n\}$ be a δ–fine division of $[a,b]$. Then each $\mathbf{I}_k \subseteq \mathbf{O}(\xi_k) \subseteq \mathbf{G}_k$, where $\mathbf{G}_k \in \mathbf{C}$. Hence

$$[a,b] = \cup_k \mathbf{I}_k \subseteq \cup_k \mathbf{G}_k.$$

The class $\{\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n\}$ is a finite subcover of $[a,b]$. □

5.0 The Intermediate Value Theorem. If f is continuous on $[a,b]$, and we have $f(a)f(b) < 0$, then there exists x_o in (a,b) such that $f(x_o) = 0$.

Proof. The proof is by contraposition. We shall assume that f is continuous and is *never* 0 on $[a,b]$, and prove that $f(a)f(b) > 0$. This inequality holds, if f has only one sign on $[a,b]$.

Let $x \in [a,b]$ and suppose that $\varepsilon = \frac{1}{2}|f(x)| > 0$. Since f is continuous at x, there is a $\delta(x) > 0$ such that if $y \in [a,b]$ and $|x - y| < \delta(x)$, then we have $|f(x) - f(y)| < \varepsilon$. Thus if $y \in [a,b]$ and $|x - y| < \delta(x)$, then

$$f(x) - \varepsilon < f(y) < f(x) + \varepsilon, \text{ or}$$

$$f(x) - \frac{1}{2}|f(x)| < f(y) < f(x) + \frac{1}{2}|f(x)|.$$

Hence for each $x \in [a,b]$, depending on whether $f(x) > 0$, or $f(x) < 0$,

(5.1) either $f > 0$, or $f < 0$ throughout $\mathbf{O}(x) \cap [a,b]$.

By Cousin's Lemma, there is a δ–fine division $D = \{([u,v];\xi)\}$ of $[a,b]$. Since $[u,v] \subseteq \mathbf{O}(\xi)$, by (5.1), it follows that f has only one sign in each subinterval $[u,v]$ in D. Since two consecutive subintervals have a common endpoint, then f has only one sign throughout $[a,b]$. □

6.0 Bolzano-Weierstrass Theorem. If **S** is a bounded, infinite set of real numbers, then **S** has an accumulation point.

Proof. Let $\mathbf{S} \subseteq [a,b]$. Assume that **S** has no accumulation point. We shall prove that the bounded set **S** is finite. Recall that x is an accumulation point of **S** if and only if $\mathbf{G} \cap \mathbf{S}$ is infinite, for each open set **G** containing x. Now let $x \in \mathbf{R}$. Since x is not an accummulation point of **S**, there is a $\delta(x) > 0$ such that $\mathbf{O}(x) \cap \mathbf{S}$ is finite. This defines a positive function δ on **R**. By Cousin's Lemma there is a δ–fine division $D = \{(\mathbf{I}_k;\xi_k) : 1 \le k \le n\}$ of $[a,b]$, such that $\xi_k \in \mathbf{I}_k \subseteq \mathbf{O}(\xi_k) \cap [a,b]$ for each k. Hence,

$$\mathbf{S} = \mathbf{S} \cap [a,b] = \cup_k(\mathbf{S} \cap \mathbf{I}_k) \subseteq \cup_k(\mathbf{O}(\xi_k) \cap \mathbf{S})$$

is finite. □

7.0 Theorem. If f is continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$.

Proof. Let $\varepsilon > 0$ and let $x \in [a,b]$. By hypothesis there is a $\delta(x) > 0$ such that for each $y \in \mathbf{O}(x) \cap [a,b]$, we have $|f(x) - f(y)| < \varepsilon/2$. Thus if u and v are in $\mathbf{O}(x) \cap [a,b]$, then

(7.1) $$|f(v) - f(u)| \le |f(v) - f(x)| + |f(x) - f(u)| < \varepsilon.$$

Let $D = \{(\mathbf{I}_k;\xi_k) : 1 \le k \le n\}$ be a δ–fine division of $[a,b]$. Let

$$\eta = \min\{\, |\mathbf{I}_k| : (\mathbf{I}_k;\xi_k) \in D \,\}.$$

Suppose $x, y \in [a,b]$ such that $|x - y| < \eta$. Then there are two possibilities.

Case 1. $x, y \in \mathbf{I}_k$, for some k. Then $x, y \in \mathbf{O}(\xi_k) \cap [a,b]$, hence by (7.1), $|f(x) - f(y)| < \varepsilon$.

Case 2. x and y respectively belong to consecutive intervals $\mathbf{I}_{k-1}$ and

$\mathbf{I}_k$ with common endpoint ξ. Then we have

$$|f(x) - f(y)| \leq |f(x) - f(\xi)| + |f(\xi) - f(y)| < 2\varepsilon.$$

Thus if x and y are in $[a,b]$ and $|x - y| < \eta$, then $|f(x) - f(y)| < 2\varepsilon$. This implies that f is uniformly continuous on $[a,b]$. □

8.0 Cantor's Intersection Theorem. Let $\{\mathbf{F}_n\}$ be a decreasing sequence of closed sets in $[a,b]$, i.e., $[a,b] \supseteq \mathbf{F}_1 \supseteq \mathbf{F}_2 \supseteq \dots$, such that $\bigcap_n F_n = \varnothing$. Then $\mathbf{F}_n = \varnothing$, for some n.

Proof outline. Let $x \in [a,b]$. Then $x \notin \mathbf{F}_n$, for some n. Since $\mathbf{F}_n$ is closed, there is a $\delta(x) > 0$ such that $\mathbf{O}(x)\cap\mathbf{F}_n = \varnothing$. This defines a positive function δ. Now apply Cousin's Lemma. □

9.0 Dini's Lemma. Let (f_n) be a sequence of continuous functions on $[a,b]$ that decreases to 0, i.e., $f_n(x) \downarrow 0$, for all x in $[a,b]$. Then the sequence (f_n) converges to 0 uniformly.

Proof. Let $\varepsilon > 0$ and let $x \in [a,b]$. Since $f_n(x) \downarrow 0$, there is a positive integer n such that $0 \leq f_n(x) < \varepsilon/2$. By the continuity of f_n, there is a positive number $\delta(x)$ such that if $y \in [a,b]$, and $|x - y| < \delta(x)$, then we have

$$|f_n(x) - f_n(y)| < \varepsilon/2.$$

Thus for each x there is an n and a $\delta(x) > 0$ such that if y is in $\mathbf{O}(x)\cap[a,b]$, then $f_n(y) < \varepsilon$. By Cousin's Lemma we can find a δ–fine division D = $\{(\mathbf{I}_k;\xi_k)\}$ of $[a,b]$. Since $\mathbf{I}_k \subseteq \mathbf{O}(\xi_k)$, it follows that for each k, there is an n_k such that if $y \in \mathbf{I}_k$, then $f_{n_k}(y) < \varepsilon$, Now let $N = \max\{n_k\}$. then we have $f_N \leq f_{n_k}$, for all k. Thus if $y \in [a,b]$, then $y \in \mathbf{I}_k$, for some k, hence $f_N(y) \leq f_{n_k}(y) < \varepsilon$. Therefore (f_n) converges to 0 uniformly. □

10.0 Remark. The next two theorems, i.e., Theorems 11.0 and 12.0, are two versions of the standard calculus theorem which says that *if the derivative of f is nonnegative throughout* $[a,b]$, *then f is nondecreasing on* $[a,b]$. (See also Theorem 13.0) With **Thomson's Lemma**, Botsko [B1] proves this theorem using lower derivatives instead of derivatives. The **lower derivative** of f at x is defined by

$$\underline{D}f(x) = \liminf \frac{f(x) - f(y)}{x - y}$$

as y tends to x. A lower derivative always exists, because it is defined in terms of the limit infimum. In Theorem 13.0, we shall rewrite Botsko's proof using the language of Cousin's Lemma.

Cousin's Lemma is also useful in extending theorems that use the 'everywhere' condition to theorems that use the 'almost everywhere' condition. In Theorem 12.0, we extend Theorem 11.0 by assuming that the derivatives exist almost everywhere.

11.0 Theorem. If $f'(x) = 0$, for all x in $[a,b]$, then f is a constant.

Proof. Let $\varepsilon > 0$ and let $x \in [a,b]$. Then there is a $\delta(x) > 0$ such that $y \in \mathbf{O}(x) \cap [a,b]$ implies $|f(x) - f(y)| \leq \varepsilon|x - y|$. Apply Cousin's Lemma. For the rest of the argument see the proof of Theorem 12.0. We can also deduce this theorem as a corollary of Theorem 13.0. □

12.0 Theorem. If $f'(x) = 0$, almost everywhere in $[a,b]$, and f is absolutely continuous, then f is constant on $[a,b]$.

Proof. Let $\varepsilon > 0$. Let $\mathbf{D} = \{ x \in [a,b] : f'(x) = 0 \}$, and let $\mathbf{E} = [a,b] \setminus \mathbf{D}$. Then for each x in $\mathbf{D}$ there exists a $\delta(x) > 0$ such that if y is in $\mathbf{O}(x) \cap [a,b]$, then $|f(x) - f(y)| \leq \varepsilon|x - y|$. This defines a function $\delta(x)$ on $\mathbf{D}$. Observe further that if $x \in \mathbf{D}$ and $x \in [u,v] \subseteq \mathbf{O}(x) \cap [a,b]$, then

$$|f(v) - f(u)| \leq |f(v) - f(x)| + |f(x) - f(u)|.$$

Therefore, we have

$$(12.1) \qquad |f(v) - f(u)| \leq \varepsilon|v - x| + \varepsilon|x - u| = \varepsilon(v - u).$$

Next, since f is absolutely continuous, there exists $\eta > 0$ such that for every (finite or infinite) sequence $([a_k, b_k])$ of disjoint subintervals of $[a,b]$,

$$(12.2) \qquad \sum_k |b_k - a_k| < \eta \text{ implies } \sum_k |f(b_k) - f(a_k)| < \varepsilon.$$

Since $\mathbf{E}$ has measure 0, there is an open set $\mathbf{G}$ of measure $|\mathbf{G}| < \eta$ such that $\mathbf{E} \subseteq \mathbf{G}$. For each x in $\mathbf{E}$ there is a $\delta(x) > 0$ such that $\mathbf{O}(x) \subseteq \mathbf{G}$. This extends the function $\delta(x)$ to all of $[a,b]$. By Cousin's Lemma, there is a δ-fine division $D_0 = \{([u,v];\xi)\}$ of $[a,b]$, where $\xi \in [u,v] \subseteq \mathbf{O}(x) \cap [a,b]$. We write $D_0 = D_1 \cup D_2$, where

$$D_1 = \{ ([u,v];\xi) \in D_0 : \xi \in \mathbf{D}\} \text{ and } D_2 = D_0 \setminus D_1.$$

In view of (12.1) and the fact that the disjoint subintervals in D_2 have total length $< \eta$, being all in **G**, it follows that

$$\begin{aligned} |f(b) - f(a)| &\leq (D_0)\sum|f(v) - f(u)| \\ &= (D_1)\sum|f(v) - f(u)| + (D_2)\sum|f(v) - f(u)| \\ &< \varepsilon(b-a) + \varepsilon = \varepsilon(b - a + 1). \end{aligned}$$

Since ε is arbitrary this implies that $f(a) = f(b)$. In the same manner we can show that if $a \leq \alpha \leq \beta \leq b$, we have $f(\alpha) = f(\beta)$. Therefore f is constant on the interval $[a,b]$. □

13.0 Theorem. If $\underline{D}f(x) \geq 0$ everywhere on an interval $[a,b]$, then f is nondecreasing on that interval.

Proof. Let $\varepsilon > 0$ and let $x \in [a,b]$. Since $\underline{D}f(x) \geq 0$, there exists a $\delta(x) > 0$ such that whenever $y \in \mathbf{O}(x) \cap [a,b]$, and $y \neq x$,

(13.1) $$(f(x) - f(y))/(x - y) \geq -\varepsilon.$$

This defines a positive function δ on $[a,b]$.

Now let $x \in [u,v] \subseteq \mathbf{O}(x) \cap [a,b]$. Then u, v are in $\mathbf{O}(x) \cap [a,b]$. Since $u \leq x \leq v$, it follows from (13.1) that

(13.2) $$f(x) - f(u) \geq -\varepsilon(x - u) \text{ and } f(v) - f(x) \geq -\varepsilon(v - x).$$

Adding the two inequalities in (13.2), we obtain

(13.3) $$f(v) - f(u) \geq -\varepsilon(v - u).$$

By Cousin's Lemma, there is a δ-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$. It follows from (13.3) that

$$\begin{aligned} f(b) - f(a) &= (D)\sum(f(v) - f(u)) \\ &\geq -\varepsilon(D)\sum(v - u) = -\varepsilon(b - a). \end{aligned}$$

Since ε is arbitrary, we have $f(b) - f(a) \geq 0$. Hence $f(b) \geq f(a)$. □

14.0 The Mean Value Theorem. If F is differentiable on $[a,b]$ and $m \leq F'(x) \leq M$ for all x in $[a,b]$, then

$$m(b-a) \leq F(b) - F(a) \leq M(b-a).$$

Proof. Let $\varepsilon > 0$ and let $x \in [a,b]$. Then there is a $\delta(x) > 0$, such that if $x \in [u,v] \subseteq \mathbf{O}(x) \cap [a,b]$, then

$$|F(v) - F(u) - F'(x)(v-u)| \leq \varepsilon |v-u|.$$

It follows from this that if $x \in [u,v] \subseteq \mathbf{O}(x)$, then

$$(m-\varepsilon)(v-u) \leq F(v) - F(u) \leq (M+\varepsilon)(v-u). \tag{14.1}$$

By Cousin's Lemma, there is a δ–fine division D = $\{([u,v];\xi)\}$ of $[a,b]$. Since $\xi \in [u,v] \subseteq \mathbf{O}(\xi)$, it follows from (14.1) that

$$(D)\sum(m-\varepsilon)(v-u) \leq (D)\sum F(v) - F(u) \leq (D)\sum(M+\varepsilon)(v-u).$$

$$\therefore\ (m-\varepsilon)(b-a) \leq F(b) - F(a) \leq (M+\varepsilon)(b-a).$$

Since ε is arbitrary the desired conclusion follows immediately. □

15.0 Definition. A function $f : [a,b] \to \mathbf{R}$ satisfies the **strong Lusin condition**, if for each $\varepsilon > 0$ and for each subset $\mathbf{E} \subseteq [a,b]$ of measure 0, there is a positive function $\delta(x)$ on **R** such that for every partial δ–fine division D = $\{([u,v];\xi)\}$ of $[a,b]$ with $\xi \in \mathbf{E}$, we have the inequality $(D)\sum|f(v) - f(u)| < \varepsilon$. Here f is called a **strong Lusin function.**

16.0 Theorem. If f is a strong Lusin function on $[a,b]$, g is non-decreasing on $[a,b]$, and $|f'(x)| \leq g'(x)$ a.e. on $[a,b]$, then we have

$$|f(b) - f(a)| \leq g(b) - g(a).$$

Proof. Let $\varepsilon > 0$ and let

$$\mathbf{D} = \{\, x \in [a,b] : |f'(x)| \leq g'(x) \,\}.$$

Then for each x in **D**, there is a positive number $\delta(x)$ such that if y is in $\mathbf{O}(x) \cap [a,b]$ and $y \neq x$, we have

$$|f(y) - f(x)| \leq |g(y) - g(x)| + \varepsilon|y-x|.$$

Thus if $x \in [u,v] \subseteq \mathbf{O}(x) \cap [a,b]$, then

$$|f(v) - f(u)| \leq g(v) - g(u) + \varepsilon(v-u). \tag{16.1}$$

On the other hand, let $\mathbf{E} = [a,b] \setminus \mathbf{D}$. Then $|\mathbf{E}| = 0$. Since f is a strong Lusin function, for each $x \in \mathbf{E}$, there exists $\delta(x) > 0$ such that for any δ–fine partial division $D = \{([u,v];\xi)\}$ of $[a,b]$ with $\xi \in \mathbf{E}$ we have

$$(16.2) \qquad (D)\sum|f(v) - f(u)| < \varepsilon.$$

Having defined δ there is, by Cousin's Lemma, a δ–fine division $D_0 = \{([u,v];\xi)\}$ of $[a,b]$. Since D_0 is δ–fine, $\xi \in [u,v] \subseteq \mathbf{O}(x)\cap[a,b]$.

Let $D_0 = D_1 \cup D_2$, a disjoint union, where

$$D_1 = \{([u,v];\xi) : \xi \in \mathbf{E}\} \text{ and}$$

$$D_2 = D_0 \setminus D_1 = \{([u,v];\xi) : \xi \in \mathbf{D}\}.$$

Hence we have from (16.1) and (16.2) that

$$\begin{aligned} |f(b) - f(a)| &= (D_1)\sum|f(v) - f(u)| + (D_2)\sum|f(v) - f(u)| \\ &< \varepsilon + (D_2)\sum(g(v) - g(u)) + (D_2)\sum\varepsilon(v - u) \\ &\leq \varepsilon + g(b) - g(a) + \varepsilon(b - a) \end{aligned}$$

Since ε is arbitrary, we have $|f(b) - f(a)| \leq g(b) - g(a)$. □

17.0 Lemma. Let f be a real-valued function on $[a,b]$ and let B be a positive real number. Suppose f is continuous on $\mathbf{X} \subseteq [a,b]$. Then there is a positive function $\delta(x)$ defined on $\mathbf{R}$ such that for each $x \in \mathbf{X}$ and for any $\mathbf{S} \subseteq \mathbf{O}(x)\cap[a,b]$, we have

$$\sup_S f - \inf_S f \leq B.$$

Proof. By continuity, for each x in $\mathbf{X}$ there is a $\delta(x) > 0$ such that if y is in $\mathbf{O}(x)\cap[a,b]$, then $f(x) - B/2 < f(y) < f(x) + B/2$. Thus if $\mathbf{S} \subseteq \mathbf{O}(x)\cap[a,b]$, then we have

$$f(x) - B/2 \leq \inf_S f \leq \sup_S f \leq f(x) + B/2. \quad \square$$

18.0 Theorem. If f is bounded on $[a,b]$ and continuous almost everywhere, then f is Riemann integrable on $[a,b]$.

Proof. There exists $M > 0$ such that $|f(x)| \leq M$ for all x in $[a,b]$. Let $\mathbf{D} = \{ x \in [a,b] : f \text{ is continuous at } x \}$ and let $\mathbf{E} = [a,b] \setminus \mathbf{D}$ Then $|\mathbf{E}| = 0$.

Let $\varepsilon > 0$ and define a positive real number B so that $2B(b - a) = \varepsilon$. Then by Lemma 17.0, there is a positive function $\delta(x)$ on $\mathbf{D}$ such that if x is in $\mathbf{D}$ and $\mathbf{S} \subseteq \mathbf{O}(x)\cap[a,b]$.

(18.1) $$\sup_S f - \inf_S f \le B.$$

On the other hand, since **E** is of measure 0, there is an open set **G** containing **E** and is of measure $|\mathbf{G}| > 0$, such that $4M|\mathbf{G}| < \varepsilon$. Then for each $x \in \mathbf{E}$ there is a $\delta(x) > 0$ such that $\mathbf{O}(x) \subseteq \mathbf{G}$. This extends the definition of δ to all of $[a,b]$.

By Cousin's Lemma, let $D_0 = \{([u,v];\, \xi)\} = \{(\mathbf{I}_k; \xi_k)\}$ be a δ-fine division of $[a,b]$. Let $D_0 = D_1 \cup D_2$ be a disjoint union, where

$$D_1 = \{([u,v];\, \xi) \in D_0 : \xi \in \mathbf{D}\} \text{ and}$$

$$D_2 = D_0 \setminus D_1 = \{([u,v];\, \xi) \in D_0 : \xi \in \mathbf{E}\}.$$

For each k, let $M_k = \sup f(x)$ and $m_k = \inf f(x)$ for all x in $\mathbf{I}_k$. If $\xi_k \in \mathbf{D}$, then it follows from (18.1) that $M_k - m_k \le B$. On the other hand, if $\xi_k \in \mathbf{E}$, then $\mathbf{I}_k \subseteq \mathbf{G}$. Hence, since the $\mathbf{I}_k$'s are non-overlapping, we have

(18.2) $$(D_2)\sum |\mathbf{I}_k| \le |\mathbf{G}| < \varepsilon/4M.$$

Since $M_k - m_k \le 2M$ for all k, the difference of the upper sum and the lower sums is

$$\begin{aligned}
\overline{S}(f,D_0) - \underline{S}(f,D_0) &= (D_1)\sum(M_k - m_k)|\mathbf{I}_k| + (D_2)\sum(M_k - m_k)|\mathbf{I}_k| \\
&< (D_1)\sum B|\mathbf{I}_k| + (D_2)\sum 2M|\mathbf{I}_k| \\
&\le B(b-a) + 2M|\mathbf{G}| \\
&< \varepsilon/2 + 2M\,(\varepsilon/4M) = \varepsilon. \qquad \text{(by (18.2))}
\end{aligned}$$

Therefore, f is Riemann integrable on $[a,b]$. □

The next result is a direct consequence of Theorem 18.0.

19.0 Theorem. If f is a continuous real-valued function on $[a,b]$, then f is Riemann integrable on $[a,b]$.

Proof outline. To prove this theorem from the definition, we must, as in Theorem 18.0, show that for some division D of $[a,b]$, we can make the difference of the upper and lower sums small, i.e., given $\varepsilon > 0$,

$$\overline{S}(f,D) - \underline{S}(f,D) = \sum_k (M_k - m_k)|\mathbf{I}_k| < \varepsilon,$$

where $M_k = \sup f(x)$ and $m_k = \inf f(x)$ for all x in $\mathbf{I}_k$. Using Lemma 17.0, we can find a positive function δ on **R** and hence a corresponding δ-fine division $D = \{(\mathbf{I}_k;\xi_k)\}$ of $[a,b]$ so that if $\xi_k \in \mathbf{I}_k \subseteq \mathbf{O}(\xi_k)\cap[a,b]$, then

$$M_k - m_k \leq \varepsilon/(b-a). \quad \square$$

Our final theorem is another generalization of Theorem 13.0. See [L, Th. 6.11] for an analogous result.

20.0 Theorem. If F is a strong Lusin function and $F'(x) \geq 0$ a.e. on an interval $[A,B]$, then F is nondecreasing on that interval.

Proof. Let a and b belong to $[A,B]$ with $a < b$. We will show that $F(a) \leq F(b)$. This will follow if we can show that given any $\varepsilon > 0$, we have $F(b) - F(a) \geq -\varepsilon(b-a)$.

Let. $\varepsilon > 0$. Let $\mathbf{D}$ be the set of all points of $[a,b]$ such that $F'(x)$ does not exist, or if it does, $F'(x) < 0$. Let $\mathbf{E} = [a,b] \setminus \mathbf{D}$. Then $|\mathbf{D}| = 0$. If $x \in \mathbf{E}$, then corresponding to the given $\varepsilon > 0$, there is a $\delta(x) > 0$ such that whenever $x \in [u,v] \subseteq \mathbf{O}(x)\cap[a,b]$, we have

$$|F(v) - F(u) - F'(x)(v-u)| \leq \varepsilon|v-u|.$$

Therefore, we have

$$F'(x)(v-u) - \varepsilon(v-u) \leq F(v) - F(u).$$

Hence, if $x \in \mathbf{E}$, there is a $\delta(x)$ such that if $x \in [u,v] \subseteq \mathbf{O}(x)\cap[a,b]$,

$$F(v) - F(u) \geq -\varepsilon(v-u). \tag{20.1}$$

On the other hand, since $|\mathbf{D}| = 0$ and F is a strong Lusin function, there is a positive function δ_1 such that for any δ_1–fine partial division $D = \{([u,v];\xi)\}$ of $[a,b]$ with $\xi \in \mathbf{D}$ we have $(D)\sum|F(v) - F(u)| < \varepsilon$.

Hence, we have

$$-\varepsilon < -(D)\sum|F(v) - F(u)| \leq (D)\sum(F(v) - F(u)). \tag{20.2}$$

Now extend the function δ to $\mathbf{R}$ by defining $\delta(x) = \delta_1(x)$ for all x not in $\mathbf{E}$. By Cousin's Lemma, let $D_0 = \{([u,v];\xi)\}$ be a δ–fine division of $[a,b]$. Hence $\xi \in [u,v] \subseteq \mathbf{O}(x)\cap[a,b]$.

Let $D_0 = D_1 \cup D_2$ be a disjoint union, where

$$D_1 = \{([u,v];\xi) : \xi \in \mathbf{D}\} \text{ and}$$

$$D_2 = D_0 \setminus D_1 = \{([u,v];\xi) : \xi \in \mathbf{E}\}.$$

Hence, we have

$$F(b) - F(a) = (D_1)\sum F(v) - F(u) + (D_2)\sum F(v) - F(u)$$
$$\geq -\varepsilon + -\varepsilon(D_2)\sum(v - u) \text{ (by (20.1) and (20.2))}$$
$$\geq -\varepsilon(b - a + 1).$$

Since ε is arbitrary, we have $F(b) - F(a) \geq 0$. Thus $F(b) \geq F(a)$. □

21.0 Remark. Unlike the Heine-Borel Theorem and the Bolzano-Weierstrass Theorem, Cousin's Lemma does not extend easily to higher dimensions or to topological spaces. This is probably the main reason why it attracted little attention. However recent developments have shown that in the real line, this versatile lemma has proved to be quite useful.

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