The Fundamental Theorem and the Cauchy Extension for Integrals in Local Systems

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Abstract

A version of the Fundamental Theorem of Calculus for Wang and Ding's integral is formulated in this paper. At the same time, it also states and proves the Cauchy Extension Theorem for the said integral as defined using Thomson's local system.

Keywords: local system, choice, S-cover, S-integral, Henstock's lemma

1 Preliminary Concepts and Results

We first give the concepts and preliminary results that we need.

Definition 1.1 Let $\mathbb{R}$ be the real line and $2^{\mathbb{R}}$ the collection of all subsets of $\mathbb{R}$. Suppose for every $x \in \mathbb{R}$, there corresponds a nonempty $S(x) \subset 2^{\mathbb{R}}$ such that

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1. $\{x\} \notin S(x)$;

2. $x \in \sigma$ whenever $\sigma \in S(x)$;

3. $\sigma_1 \in S(x)$ and $\sigma_1 \subset \sigma_2$ implies that $\sigma_2 \in S(x)$;

4. $\sigma \in S(x)$ and $\delta > 0$ implies that $(x - \delta, x + \delta) \cap \sigma \in S(x)$.

Then $S = \{S(x) : x \in \mathbb{R}\}$ is called a **local system**. S is said to be **bilateral** if every $\sigma \in S(x)$ contains points on both sides of x. It is **filtering** if for every $x \in \mathbb{R}$, $\sigma_1 \cap \sigma_2 \in S(x)$ whenever $\sigma_1, \sigma_2 \in S(x)$. Further, it is said to have the **intersection condition** if for every collection of sets $\{\sigma(x) : x \in \mathbb{R}\}$, called a **choice from** S, where $\sigma(x) \cap S(x)$, there is a positive function δ such that if $0 < y - x < \min\{\delta(x), \delta(y)\}$, then $\sigma(x) \cap \sigma(y) \cap [x, y] \neq \varnothing$.

Definition 1.2 A family C of finite closed intervals is called an S-**cover** of $\mathbb{R}$ if for every $x \in \mathbb{R}$, the set

$$\sigma(x) = \{y : y = x, \text{ or } y > x \text{ and } [x, y] \in C, \text{ or } y < x \text{ and } [y, x] \in C\}$$

belongs to $S(x)$.

The following results are proved in [1].

Lemma 1.3 *A family C of finite closed intervals is an S-cover of $\mathbb{R}$ if and only if there exists a choice $\{\sigma(x) : x \in \mathbb{R}\}$ of S such that*

(a) if $y \in \sigma(x)$ and $y > x$, then $[x, y] \in C$ and

(b) if $y \in \sigma(x)$ and $y < x$, then $[y, x] \in C$

Lemma 1.4 *Let $\{\sigma(x) : x \in \mathbb{R}\}$ be choice from S and*

$$C = \{[u, v] \subset \mathbb{R} : u \in \tau(v) \text{ or } v \in \tau(u)\}.$$

Then C is an S-cover of $\mathbb{R}$.

In view of Lemma 1.3, it follows that every S-cover C of $\mathbb{R}$ has a corresponding choice from S. Conversely, every choice from S can be used to define an S-cover for $\mathbb{R}$. These facts will be frequently used in the next section.

Definition 1.5 Let $S = \{S(x) : x \in \mathbb{R}\}$ be a local system which is bilateral and has the intersection condition and C an S-cover for $\mathbb{R}$. A tagged division $D = \{([u, v]; \xi)\}$ is called a **C-partition** of an interval $[a, b]$ if each $[u, v]$ belongs to C. The associated or tag point is $\xi = u$ if $v \in \sigma(u)$ or $\xi = v$ if $u \in \sigma(v)$ or either one when both occur, where $\{\sigma(x) : x \in \mathbb{R}\}$ is the choice from S corresponding to C.

The following result guarantees the existence of a C-partition of an interval for every given S-cover C of R. See [4] for its proof.

Theorem 1.6 (Thomson's Lemma) *Let $S = \{S(x) : x \in \mathbb{R}\}$ be a local system which is bilateral and has the intersection condition. If C is an S-cover of $\mathbb{R}$, then there is a C-partition of any interval $[a, b]$.*

Henceforth, $S = \{S(x) : x \in \mathbb{R}\}$ is a fixed local system that is bilateral, filtering and satisfies the intersection condition.

Definition 1.7 The real number L is the *S-limit* of a function $F : [a,b] \to \mathbb{R}$ as $x \to \xi$, if for every $\epsilon > 0$, there exists a $\sigma(\xi) \in S(\xi)$ such that $|F(x) - L| < \epsilon$ whenever $x \in \sigma(\xi) \cap [a,b]$ and $x \neq \xi$. In this case, we write

$$S\text{-}\lim_{x \to \xi} F(x) = L .$$

Theorem 1.8 [1] If $\lim_{x \to \xi} F(x) = L$, then $S\text{-}\lim_{x \to \xi} F(x) = L$.

Definition 1.9 A function $F : [a,b] \to \mathbb{R}$ is *S-differentiable* at $\xi \in [a,b]$ if

$$S\text{-}\lim_{x \to \xi} \frac{F(x) - F(\xi)}{x - \xi} \quad \text{exists.}$$

We denote this limit by $SDF(\xi)$ and call it the *S-derivative* of F at ξ. F is *S-differentiable* on $[a,b]$ if it is S-differentiable at every point in $[a,b]$.

Corollary 1.10 *If $F : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$, then it is S-differentiable there. Moreover, $F'(x) = SDF(x)$ for every $x \in [a,b]$.*

Proof: This follows from Definition 1.9 and Theorem 1.8. □

Definition 1.11 A function $F : [a,b] \to \mathbb{R}$ is *S-continuous* at $\xi \in [a,b]$ if for every $\epsilon > 0$, there exists $\sigma(\xi) \in S(\xi)$ such that $|F(x) - F(\xi)| < \epsilon$ whenever $x \in \sigma(\xi) \cap [a,b]$. F is *S-continuous* on $[a,b]$ if it is continuous at every point in $[a,b]$.

Theorem 1.12 *If $F : [a,b] \to \mathbb{R}$ is S-differentiable on $[a,b]$, then it is S-continuous there.*

Proof: Let $\xi \in [a,b]$ and let $\epsilon > 0$. By assumption, corresponding to $\epsilon/(b-a)$, there exists a $\sigma(\xi) \in S(\xi)$ such that

$$|F(x) - F(\xi) - SDF(\xi)(x-\xi)| < \epsilon$$

whenever $x \in \sigma(\xi) \cap [a,b]$ and $x \neq \xi$. Set

$$\tau(\xi) = \sigma(\xi) \cap (\xi - \delta, \xi + \delta) ,$$

where

$$\delta = \epsilon/(1 + |SDF(\xi)|) .$$

By Property (4) in Definition 1.1, it follows that $\tau(\xi) \in S(\xi)$. Now, if $x \in \tau(\xi) \cap [a,b]$, then we have

$$\begin{aligned} |F(x) - F(\xi)| &\leq |F(x) - F(\xi) - SDF(\xi)(x-\xi)| + |SDF(\xi)||x-\xi| \\ &< 2\epsilon . \end{aligned}$$

This shows that F is S-continuous at ξ. Therefore F is S-continuous on $[a,b]$. □

Lemma 1.13 *If* $\xi \in \mathbb{R}$ *and* $\delta > 0$, *then* $(\xi - \delta, \xi + \delta) \in S(\xi)$.

Proof: By Property (3) of Definition 1.1, the real line $\mathbb{R}$ is in $S(\xi)$. Thus, by Property (iv),

$$(\xi - \delta, \xi + \delta) = (\xi - \delta, \xi + \delta) \cap R \in S(\xi) .$$ □

Theorem 1.14 *If* $F : [a,b] \to \mathbb{R}$ *is continuous on* $[a,b]$, *then it is* S-*continuous there.*

Proof: Let $\xi \in [a, b]$ and let $\epsilon > 0$. By continuity of F at ξ, there exists a $\delta > 0$ such that $|F(x) - F(\xi)| < \epsilon$ whenever $x \in (\xi - \delta, \xi + \delta) \cap [a, b]$. Put $\sigma(\xi) = (\xi - \delta, \xi + \delta)$. Then $\sigma(\xi) \in S(\xi)$ by Lemma 1.13. Further, if $x \in \sigma(\xi) \cap [a, b]$, then $|F(x) - F(\xi)| < \epsilon$. This proves the theorem. □

Definition 1.15 A real-valued function f defined on $[a, b]$ is S**-integrable** to the number A if for every $\epsilon > 0$ there exists an S-cover C of $\mathbb{R}$ such that for any C-partition $D = \{([u, v]; \xi)\}$ of $[a, b]$,

$$\left|(D) \sum f(\xi)(v - u) - A\right| < \epsilon.$$

The S-integral A of f, if it exists, is unique. In symbols we write

$$(S) \int_a^b f(t)\, dt = (S) \int_a^b f = A.$$

It is shown in [1] and [5] that the set of all S-integrable real-valued functions defined on $[a, b]$ is a real vector space and that S-integrability over the whole interval implies S-integrability on every subinterval. Further, the following results were also proved.

Theorem 1.16 *If $f(x) = 0$ almost everywhere in $[a, b]$ then f is S-integrable to zero on $[a, b]$.*

Corollary 1.17 *Let f and g be real-valued functions on $[a, b]$. If f is S-integrable on $[a, b]$ and $f = g$ almost everywhere on $[a, b]$, then g is also S-integrable on $[a, b]$. Moreover,*

$$(S) \int_a^b f = (S) \int_a^b g.$$

Theorem 1.18 (Henstock's Lemma) *If $f : [a, b] \to \mathbb{R}$ is S-integrable on $[a, b]$ with S-primitive F defined by*

$$F(x) = (S) \int_a^x SDF(t)\, dt = F(x) - F(a)$$

for every $x \in [a, b]$, then for every $\epsilon > 0$, there exists an S-cover C of $\mathbb{R}$ such that for any C-partition $D = \{([u, v]; \xi)\}$ of $[a, b]$,

$$\left|(D) \sum f(\xi)(v - u) - F(v) + F(u)\right| < \epsilon.$$

2 Main Results

The first result says that the S-integral recovers a function (an S-continuous function) from its S-derivative.

Theorem 2.1 (Fundamental Theorem) *Let $F : [a, b] \to \mathbb{R}$ be S-continuous on $[a, b]$. If F is S-differentiable nearly everywhere on $[a, b]$(i.e. except for a countable set), then SDF is S-integrable on $[a, b]$ and*

$$(S) \int_a^x SDF(t)\, dt = F(x) - F(a)$$

for every $x \in [a, b]$.

Proof: Let $E = \{x_n \in [a, b] : SDF(x_n)$ does not exist$\}$. Define $G : [a, b] \to \mathbb{R}$ by

$$G(x) = SDF(x)$$

if $x \in [a, b] \setminus E$ and $G(x) = 0$ if $x \in E$. Then $G(x) = SDF(x)$ nearly everywhere. We shall show that G is S-integrable on $[a, b]$.

So, let $\epsilon > 0$. By S-differentiability of F on $[a,b] \setminus E$, there exists $\sigma(x) \in S(x)$ for each $x \in [a,b] \setminus E$ such that if $y \in \sigma(x) \cap [a,b]$ and $y \neq x$, then

$$|F(y) - F(x) - SDF(x)(y-x)| < \epsilon \cdot |y-x|.$$

For $x = x_n$, use S-continuity of F at x to choose $\tau(x) \in S(x)$ so that

$$y \in \tau(x) \cap [a,b] \text{ implies } |F(y) - F(x)| < \epsilon \cdot 2^{-n}.$$

Next, define $v(x) = \sigma(x)$ if $x \in [a,b] \setminus E$, $v(x) = \tau(x)$ if $x \in E$, and $v(x) = \mathbb{R}$ if otherwise. Then $v(x) \in S(x)$ for every real number x and $M = \{v(x) : x \in \mathbb{R}\}$ is a choice from S. Let C be an S-cover of $\mathbb{R}$ corresponding to M and $D = \{([u,v];\xi)\}$ a C-partition of $[a,b]$. Write $D = D_1 \cup D_2$, where $D_1 = \{([u,v];\xi) \in D : \xi \in E\}$ and $D_2 = \{([u,v];\xi) \in D : \xi \in [a,b] \setminus E\}$. Then

$$\begin{aligned}
&|(D)\sum G(\xi)(v-u) - F(b) + F(a)| \\
&\quad \le |(D_1)\sum G(\xi)(v-u) - F(b) + F(a)| \\
&\qquad\qquad + |(D_2)\sum G(\xi)(v-u) - F(b) + F(a)| \\
&\quad \le (D_1)\sum |G(\xi)(v-u) - F(v) + F(u)| + (D_2)\sum |F(v) - F(u)| \\
&\quad < \epsilon \cdot (b-a) + \sum_{n=1}^{\infty} \epsilon \cdot 2^{-n} \\
&\quad = (b-a+1)\epsilon.
\end{aligned}$$

This shows that G is S-integrable on $[a,b]$ and

$$(S)\int_a^b G(t)\, dt = F(b) - F(a).$$

Therefore, by Theorem 1.18, SDF is S-integrable and

$$(S)\int_a^b SDF(t)\, dt = F(b) - F(a).$$

The above argument can be applied to any interval $[a, x]$ $\forall x \in [a, b]$. □

Lemma 2.2 *Suppose that* $(S) \int_a^b f(t)\, dt$ *exists for every* $x \in (a, b)$. *Then for every* $\epsilon > 0$, *there exists an* S*-cover* C *of* $\mathbb{R}$ *such that if* $c \in (a, b)$ *and* $D = \{([u, v]; \xi)\}$ *is a* C*-partition of* $[c, b]$, *then*

$$\left|(D) \sum f(\xi)(v - u) - (S) \int_a^b f\right| < \epsilon.$$

Proof: Let $\{a_n\}$ be a decreasing sequence in (a, b) that converges to a. Set $a_0 = b$ and let $\epsilon > 0$. For each n, there exists an S-cover C_n of $\mathbb{R}$ such that if $D = \{([u, v]; \xi)\}$ is a C_n-partition of $[a_n, a_{n-1}]$, we have

$$\left|(D) \sum f(\xi)(v - u) - (S) \int_u^v f\right| < \epsilon \cdot 2^{-n}.$$

For each n, let $\{\sigma_n(x) : x \in \mathbb{R}\}$ be a choice from S corresponding to C_n. Define

$$\tau(x) = \begin{cases} \sigma_1(x) \cap (a_1 + \frac{3}{4}(a_0 - a_1), a_0 + \frac{1}{4}(a_0 - a_1)) & \text{, if } x = a_0 = b \\ \sigma_n(x) \cap (a_n, a_{n-1}) & \text{, if } a_n < x < a_{n-1} \text{ for some } n \\ \sigma_n(x) \cap (a_{n+1}, a_{n-1}) & \text{, if } x = a_n \text{ for some } n \\ \mathbb{R} & \text{, elsewhere.} \end{cases}$$

Then $\{\tau(x) : x \in \mathbb{R}\}$ is a choice from S. Let

$$C = \{[u, v] \subset \mathbb{R} : u \in \tau(v) \text{ or } v \in \tau(u)\}.$$

Then by Lemma 1.2, C is an S-cover of $\mathbb{R}$. Let $c \in (a, b)$ and $D = \{([u, v]; \xi)\}$ a C-partition of $[c, b]$. Choose an integer m such that

$$a_{m+1} < c < a_m.$$

Note that by definition of $\tau(x)$, a_n is an associated point of D for every $n = 1, 2, \ldots, m$. Also, for every interval-point pair $([u, v]; \xi)$ in D, there exists some $n = 1, 2, \ldots, m+1$ such that $[u, v] \subset [a_n, a_{n-1}]$. For such an n, let D_n denote the set of pairs $([u, v]; \xi)$ in D such that $[u, v] \subset [a_n, a_{n-1}]$. Note that D_n is a partial C_n-partition of $[a_n, a_{n-1}]$. Also, observe that

$$D_{m+1} = D \setminus \bigcup_{n=1}^{m} D_n$$

is a partial C_{m+1}-partition of $[a_{m+1}, a_m]$. Thus,

$$\begin{aligned}
\left|(D)\sum f(\xi)(v-u) - (S)\int_c^b f\right| &= \left|(D)\sum \left\{f(\xi)(v-u) - (S)\int_u^v f\right\}\right| \\
&\le \sum_{n=1}^{m+1} \left|(D_n)\sum \left\{f(\xi)(v-u) - (S)\int_u^v f\right\}\right| \\
&< \sum_{n=1}^{\infty} \epsilon \cdot 2^{-n} = \epsilon\,.
\end{aligned}$$

This completes the proof of the theorem. □

We now prove the **Cauchy Extension Theorem.**

Theorem 2.3 (Cauchy Extension) *(i) If $f : [a, b] \longrightarrow \mathbb{R}$ is S-integrable on each interval $[x, b]$ for every $x \in (a, b]$ and $(S)\lim_{x \to a} \int_x^b f$ exists, then f is S-integrable on $[a, b]$ and*

$$(S)\int_a^b f = (S)\lim_{x\to a}\int_x^b f.$$

(ii) If $f : [a, b] \longrightarrow \mathbb{R}$ is S-integrable on each interval $[a, x]$ for every $x \in [a, b)$ and $(S)\lim_{x \to b} \int_a^x f$ exists, then f is S-integrable on $[a, b]$ and

$$(S)\int_a^b f = (S)\lim_{x\to b}\int_a^x f.$$

Proof: Let

$$L = (S)\lim_{x\to a}\int_x^b f$$

and let $\epsilon > 0$. By Lemma 2.2, there exists an S-cover $C*$ of $\mathbb{R}$ such that if $x \in (a, b)$ and $D = \{([u, v]; \xi)\}$ is a $C*$-partition of $[x, b]$, then

$$\left|(D) \sum f(\xi)(v-u) - (S) \int_x^b f\right| < \epsilon.$$

Let $\{\sigma * (x) : \ x \in \mathbb{R}\}$ be a choice corresponding to $C*$. Since $(S) \lim_{x \to a} \int_x^b f$ exists, there exists $\sigma(a) \in S(a)$ such that whenever $x \in \sigma(a) \cap [a, b]$, we have

$$\left|(S) \int_x^b f - L\right| < \epsilon.$$

Define

$$\tau(x) = \begin{cases} \sigma * (x) \cap (a, x+\epsilon) & , \text{ if } x \in (a, b] \\ \sigma(a) \cap (a - \frac{\epsilon}{1+|f(a)|}, a + \frac{\epsilon}{1+|f(a)|}) & , \text{ if } x = a \\ \mathbb{R} & , \text{ elsewhere} \end{cases}$$

Let $C = \{[u, v] \subset \mathbb{R} : \ u \in \tau(v) \text{ or } v \in \tau(u)\}$. Then by Lemma 1.2, C is an S-cover of $\mathbb{R}$. Let $D = \{([u, v]; \xi)\}$ a C-partition of $[a, b]$. There exists a c such that $([a, c]; \xi)$ is in D. Since $a \notin \tau(c)$, it follows that $\xi = a$. Write $D = D_1 \cup D_2$, where $D_2 = \{([a, c]; a)\}$ and $D_1 = D \setminus D_2$. Then

$$\begin{aligned} \left|(D) \sum f(\xi)(v-u) - L\right| &= \left|(D_1) \sum f(\xi)(v-u) + f(a)(c-a) - L\right| \\ &\leq \left|(D_1) \sum f(\xi)(v-u) - (S) \int_c^b f\right| \\ &\quad + \left|(S) \int_c^b f - L\right| + |f(a)|(c-a) \\ &< \epsilon + \epsilon + \epsilon \\ &= 3\epsilon. \end{aligned}$$

This proves that f is S-integrable on $[a, b]$. Part (ii) can be proved using a similar argument. □

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