On d-Unit Graphs in $\mathbb{R}^n$

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Abstract

In this paper, the unit dimension of a graph with respect to a metric d on the space $\mathbb{R}^n$ is defined. This generalizes the concept of the Euclidean unit dimension of a graph in the space $\mathbb{R}^n$ as defined by various authors in [1], [2], [3], and [4]. It is also shown that two equivalent metrics in $\mathbb{R}^n$ may yield different unit dimensions of some complete graphs.

Keywords: Euclidean, unit, dimension, graph, metric, representation

1 Introduction

The Euclidean unit dimension of graphs had been defined and studied by various authors in [1], [2], [3], and [4]. In this paper, we make an initial step to generalize the said concept by using an arbitrary metric d on the space $\mathbb{R}^n$. We shall consider two particular metrics on $\mathbb{R}^n$ which are equivalent to the Euclidean (or usual) metric on $\mathbb{R}^n$ and see that these metrics yield different unit dimensions of some complete graphs.

Throughout this paper, any graph $G = (V(G), E(G))$, where $V(G)$, and $E(G)$ are, respectively, the vertex set and the edge set of G, is a simple graph

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and d is a metric on $\mathbb{R}^n$.

Definition 1.1 A graph $G = (V(G), E(G))$ is a *d-unit graph* in $\mathbb{R}^k$ if there exists a one-to-one mapping $\phi : V(G) \to \mathbb{R}^k$ such that $d(\phi(x), \phi(y)) = 1$ whenever $[x, y] \in E(G)$. Such a mapping is called a *d-unit representation* or *d-unit embedding* of G in $\mathbb{R}^k$.

Note that if ϕ is a d-unit embedding of G in $\mathbb{R}^k$, then we may look at G as the graph $\phi(G)$, where $V(\phi(G)) = \{\phi(x) : x \in V(G)\}$ and $E(G) = \{[\phi(x), \phi(y)] : [x, y] \in E(G)\}$, in $\mathbb{R}^k$.

Remark 1.2 *For each $n \geq 1$, there exists a metric d^* on $\mathbb{R}^n$ such that if $G = (V(G), E(G))$ is a graph with $E(G) \neq \varnothing$, then G has no d^*-unit representation on $\mathbb{R}^n$.*

To see this, let $n \geq 1$ and consider the trivial metric d^* on $\mathbb{R}^n$ defined by $d^*(x, y) = 1$ if $x = y$ and $d^*(x, y) = 0$ otherwise. If $E(G) \neq \varnothing$, then G has at least two distinct vertices v_1 and v_2 such that v_1v_2 is in $E(G)$. If ϕ is a one-to-one mapping from $V(G)$ into $\mathbb{R}^n$, then $\phi(v_1) \neq \phi(v_2)$. It follows that $d^*(\phi(v_1), \phi(v_2)) \neq 1$. This implies that G cannot have a d^*-unit embedding in $\mathbb{R}^n$.

Definition 1.3 Let d be a metric on $\mathbb{R}^n$. The *dimension* of G with respect to the metric d, denoted by d-$Dim(G)$ is given by

$$d\text{-}Dim(G) = \min\{n \in \mathbb{N} \cup \{0\} : G \text{ has a } d\text{-unit embedding in } \mathbb{R}^n\}.$$

Notice that if there exists a natural number k such that G has a d-unit representation in $\mathbb{R}^k$, then by the Well Ordering Principle, d-$Dim(G)$ exists.

On the other hand, if G has no d-unit embedding in $\mathbb{R}^n$ for every non-negative integer n, then d-$Dim(G)$ does not exist. This brings us to the next remark which easily follows from the first and Definition 1.3.

Remark 1.4 *If $G = (V(G), E(G))$ is a graph with $E(G) \neq \varnothing$ and d^* is the trivial metric on $\mathbb{R}^n$, then d^*-$Dim(G)$ does not exist.*

Theorem 1.5 *If d_1 is the usual metric on $\mathbb{R}^n$ (for all n) and $G = (V(G), E(G))$ is a graph of order k, then d_1-$Dim(G)$ exists.*

Proof: For convenience, we let $V(G) = \{1, 2, 3, \ldots, k\}$. For each $j \in V(G)$, let $p_j = (0, 0, \ldots, \frac{1}{\sqrt{2}}, 0, \ldots, 0)$ be in $\mathbb{R}^k$, where the jth component is $\frac{1}{\sqrt{2}}$ and 0 elsewhere. Clearly, $d_1(p_i, p_j) = 1$ for all pairs (i, j) with $i \neq j$. Define $\phi : V(G) \to \mathbb{R}^k$ by $\phi(j) = p_j$. Then ϕ is a unit embedding of G in $\mathbb{R}^k$. By Definition 1.3, it follows that d_1-$Dim(G)$ exists.

Theorem 1.6 *Let G be a graph and H a subgraph of G. If d-$Dim(G)$ exists, then d-$Dim(H)$ exists and d-$Dim(H) \leq d$-$Dim(G)$.*

Proof: Suppose d-$Dim(G) = n$. Let $\phi : V(G) \to \mathbb{R}^n$ be a unit embedding of G in $\mathbb{R}^n$. Since H is a subgraph of G, $V(H) \subseteq V(G)$. Let $\alpha : V(H) \to \mathbb{R}^n$ be the restriction of ϕ to $V(H)$, i.e., $\alpha = \phi_{|V(H)}$ (the restriction of ϕ to $V(H)$). Then α is a d-unit embedding of H in $\mathbb{R}^n$. By Definition 1.3, d-$Dim(H)$ exists and d-$Dim(H) \leq n = d$-$Dim(G)$. □

Theorem 1.7 *If H and G are isomorphic graphs and at least one of them has a d-unit embedding in $\mathbb{R}^k$ for some k, then d-$Dim(H)$ and d-$Dim(G)$ both exist and are equal.*

Proof: We may suppose without loss of generality that H has a d-unit embedding $\phi : V(H) \to \mathbb{R}^k$ for some k. Since H and G are isomorphic, there exists a bijective mapping $\alpha : V(G) \to V(H)$ which preserves adjacency. The composition mapping $\phi \circ \alpha : V(G) \to \mathbb{R}^k$ is a bijective mapping. Also, since

$$d((\phi \circ \alpha)(u), (\phi \circ \alpha)(v)) = d(\phi(\alpha(u)), \phi(\alpha(v)) = 1$$

for all distinct vertices u and v in G, it follows that $\phi \circ \alpha$ is a unit embedding of G in $\mathbb{R}^k$. By Definition 1.3, d-$Dim(H)$ and d-$Dim(G)$ both exist.

Now, let d-$Dim(H) = q$ and d-$Dim(G) = p$. Then H has unit embedding in $\mathbb{R}^q$. As shown above, G has also a unit embedding in $\mathbb{R}^q$. By Definition 1.3, $p \leq q$. Interchanging the roles of H and G in the above proof yields the inequality $q \leq p$. Thus, d-$Dim(H) = d$-$Dim(G)$. □

Next, we show that equivalent metrics on $\mathbb{R}^n$ (for all n) may induce different dimensions of a certain graph. In what follows, d_1 is the usual (Euclidean) metric on $\mathbb{R}^n$, d_2 and d_3 are the metrics on $\mathbb{R}^n$ defined by

$$d_2(x, y) = \max\{|x_i - x_j| : i = 1, 2, 3, \ldots, n\},$$

$$d_3(x, y) = \sum_{i=1}^{n} |x_i - x_j| \ ,$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

Theorem 1.8 **([1],[3])** *For all $n \geq 1$, $d_1(K_n) = n - 1$.*

Lemma 1.9 *Let $n \geq 1$ and $\phi : V(K_n) \to \mathbb{R}^k$ be a d_2-unit embedding of K_n in $\mathbb{R}^k$. Then there exists a d_2-unit embedding of K_n in $\mathbb{R}^k$ such that the images are points whose components consist of zeros and ones only.*

Proof: Let $V(K_n) = \{1, 2, \ldots, n\}$ and suppose that for each $j \in V(Kn)$, $\phi(j) = p_j = (\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_k})$. By definition of d_2, we may assume that the components of $\phi(j)$ are all integers for each $j \in V(K_n)$. Define $\pi : V(K_n) \rightarrow \mathbb{R}^k$ by $\pi(j) = q_j = (\beta_{j_1}(mod\ 2), \beta_{j_2}(mod\ 2), \ldots, \beta_{j_k}(mod\ 2))$. Clearly, π is a well-defined function. Let $r, s \in V(K_n)$ and suppose that $\pi(r) = \pi(s)$. Then $\beta_{rm}(mod\ 2) = \beta_{sm}(mod\ 2)$ for all $m = 1, 2, \ldots, k$. This means that for all m, 2 divides $|\beta_{rm} - \beta_{sm}|$. If $r \neq s$, then since ϕ is a d_2-unit embedding, $d_2(p_r, p_s) = 1$, i.e., there exists a natural number t, where $1 \leq t \leq k$, such that $|\beta_{rt} - \beta_{st}| = 1$. This is contrary to an earlier statement. Therefore $r = s$ and hence, π is one-to-one.

Further, if $r \neq s$, then there exists a natural number t, where $1 \leq t \leq k$, such that $|\beta_{rt} - \beta_{st}| = 1$. Consequently, $|\beta_{rt}(mod\ 2) - \beta_{st}(mod\ 2)| = 1$ and $d_2(q_r, q_s) = 1$. Thus, π is a desired d_2-unit embedding of K_n in $\mathbb{R}^k$. □

Lemma 1.10 *Let $n, k \in \mathbb{N}$ be such that $n \leq 2^k$. Then there exists a d_2-unit embedding of K_n in $\mathbb{R}^k$ the images of which are in S, where S is the subset of $\mathbb{R}^k$ consisting of points $(\beta_1, \beta_2, \ldots, \beta_k)$, where β_j is 0 or 1 for all $j = 1, 2, \ldots, k$.*

Proof: Using a combinatorial technique, it can be shown that S has 2^k elements. Also, if x and y are distinct elements of S, then $d_2(x, y) = 1$. Therefore, since $n \leq 2^k$, a d_2-unit embedding can be constructed. □

Corollary 1.11 *If $G = (V(G), E(G))$ is a graph of order n, then d_2-$Dim(G)$ exists.*

Proof: Suppose the order of G is n. Since $n \leq 2^n$, K_n has a d_2-unit embedding in $\mathbb{R}^n$ by Lemma 1.10. By Definition 1.3, d_2-$Dim(K_n)$ exists. By Theorem 1.7, since G is a subgraph of K_n, d_2-$Dim(G)$ exists.

□

Theorem 1.12 *For all* $n \geq 1$,

$$d_2\text{-}Dim(K_n) = \min\{m \in \mathbb{N} : n \leq 2^m\}.$$

Proof: Suppose d_2-$Dim(K_n) = k$. Then K_n has a d_2-unit embedding in $\mathbb{R}^k$. By Lemma 1.9, we must have $n \leq 2^k$. Hence, $k \geq r = \min\{m \in \mathbb{N} : n \leq 2^m\}$. Since $n \leq 2^r$, K_n has a d_2-unit embedding in $\mathbb{R}^r$ whose images are in S (the set in Lemma 1.10) by Lemma 1.10. Hence, by Definition 1.3, d_2-$Dim(K_n) = k \leq r$.

□

Corollary 1.13 *Let* $n \in \mathbb{N} \cup \{0\}$ *and* $m \in \mathbb{N}$ *with* $0 < m \leq 2^n$. *Then*

$$d_2\text{-}Dim(K_{2^n}) = n \quad and \quad d_2\text{-}Dim(K_{2^n+m}) = n + 1.$$

Proof: Since $n = \min\{m \in \mathbb{N} : V(K_{2^n}) = 2^n \leq 2^m\}$, the first result follows from Theorem 1.12. Also, since $0 < m \leq 2^n$, it follows that $2^n < 2^n + m = |K_{2^n+m}| \leq 2^{n+1}$. Thus, by Theorem 1.12, d_2-$Dim(K_{2^n+m}) = n+1$.

□

Example 1.14 Since $32 = 2^5$ and $7 = 2^2+3$, it follows that d_2-$Dim(K_{32}) = 5$ and $d2$-$Dim(K_7) = 3$. Note that by Theorem 1.8, $d1$-$Dim(K_{32}) = 31$ and $d1$-$Dim(K_7) = 6$.

Theorem 1.15 *Let* $n, k \in \mathbb{N}$ *be such that* $n \leq 2^k$. *Then there exists a* d_3*-unit embedding of* K_n *in* $\mathbb{R}^k$ *and* d_3-$Dim(K - n)$ *exists. Further,*

$$d_3\text{-}Dim(K_n) \leq \min\{k \in \mathbb{N} : n \leq 2^k\}.$$

Proof: Let $n, k \in N$ be such that $n \leq 2k$. Denote by T_1 the set consisting of all points $(0, \ldots, 0, 1/2, 0, \ldots, 0)$ in $\mathbb{R}^k$, where the jth component is $1/2$ and 0 elsewhere, and by T_2 the set of all points $-p$, where $p \in T_1$. Then the set $T = T_1 \cup T_2$ has $2k$ distinct elements. Therefore, since $n \leq 2k$, we can construct a one-to-one function φ from $V(K_n)$ into T. Further, since $d_3(p_1, p_2) = 1$ for any two distinct elements p_1 and p_2 of T, it follows that the function φ is a d_3-unit embedding of K_n in $\mathbb{R}^k$. Consequently, by Definition 1.3, d_3-$Dim(K_n)$ exists and d_3-$Dim(K_n) \leq \min\{k \in \mathbb{N} : n \leq 2k\}$. □

Corollary 1.16 *If $G = (V(G), E(G))$ is a graph of order n, then d_3-$Dim(G)$ exists.*

Proof: Suppose the order of G is n. By Theorem 1.15, d_3-$Dim(K_n)$ exists. By Theorem 1.7, since G is a subgraph of K_n, $d3$-$Dim(G)$ also exists. □

Corollary 1.17 *Let $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$ with $0 < m \leq 2n$. Then*

$$d_3\text{-}Dim(K_{2n}) \leq n$$

$$d_3\text{-}Dim(K_{2n+m}) \leq n + 1.$$

Proof: Since $n = \min\{m \in \mathbb{N} : |V(K_{2n})| = 2n \leq 2m\}$, the first result follows from Theorem 1.15. Also, since $0 < m \leq 2n$, $2n < 2n + m = |K_{2n+m}| \leq 2(n+1)$. Thus, by Theorem 1.15, d_3-$Dim(K_{2n+m}) = n + 1$. □

Example 1.18 Since $32 = 2(16)$ and $7 = 2(3) + 1$, it follows that d_3-$Dim(K_{32}) \leq 16$ and d_3-$Dim(K_7) = 4$.

Remark 1.19 *The upper bounds obtained in Theorem 1.15 and Corollary 1.17 are best possible. This author conjectured that the given bounds are the exact values of the dimensions.*

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