A Remark on the Axiom of Archimedes

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If we remove water from a bathtub one spoonful at a time, sooner or later, the bathtub will run out of water. The inevitable consequence is the essence of the so-called Axiom of Archimedes. This simple property of real numbers is one of the most useful tools of elementary analysis. The following theorem gives three equivalent formulations of this property. The proof is quite straightforward.

Theorem 1. The following statements are equivalent: (i) If $a, b \in \mathbf{R}$ with $a > 0$, then there is a positive integer n such that $na > b$;

(ii) If $b \in \mathbf{R}$, there is a positive integer n such that $n > b$;

(iii) The set $\mathbf{Z}^+$ of positive integers is not bounded.

Proof. (i) $\Rightarrow$ (ii): Let $a = 1$ in (i).

(ii) $\Rightarrow$ (iii): Let $b \in \mathbf{R}$. Then by (ii), there exists a positive integer $n > b$. Hence b is not an upper bound for the set $\mathbf{Z}^+$ of positive integers. Because b is arbitrary, this implies that $\mathbf{Z}^+$ is not bounded above. Hence $\mathbf{Z}^+$ is not bounded.

(iii) $\Rightarrow$ (i): Let a and b be real numbers, where $a > 0$. Since $\mathbf{Z}^+$ is not bounded (although it is bounded below, for example, by 1 or by 0), then $\mathbf{Z}^+$ is not bounded above. Hence b/a is not an upper bound for $\mathbf{Z}^+$. This means that there exists a positive integer $n > b/a$. This implies (i). □

Parts (i), (ii), and (iii) above are stated as separate theorems with short proofs in Apostol's book [A; pp. 10-11]. Apostol first deduced (iii) from the Completeness Axiom; then he proved that (iii) implies (ii), and (ii) implies (i). But Apostol did not complete the cycle by proving that (i) implies (iii). This would have proved equivalence.

Part (i) is known in the literature as the **Axiom of Archimedes** or the Archimedian property for **R**. It is usually deduced from the Completeness Axiom, which says that *a nonempty set of real numbers which is bounded*

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above has a least upper bound. Most authors, e.g., see [R; 4.6] and [PZ; Th. 1.8], do this by proving part (i) directly. Their proofs are a little complicated. Others, e.g., see [G; Th. 0.17], prefer to first prove part (ii), which is easier than part (i).

However the idea of proving (iii) first, appears to be the simplest among these alternatives. For completeness let us present a detailed proof of (iii) based on a similar proof given by Apostol in [A].

Theorem 2. The set $\mathbf{Z}^+$ of positive integers is not bounded.

Proof. To get a contradiction, suppose $\mathbf{Z}^+$ is bounded above. Then by the Completeness Axiom, $\mathbf{Z}^+$ has a least upper bound, say σ. Hence $(\sigma - 1)$ is not an upper bound for $\mathbf{Z}^+$. Thus there exists $n \in \mathbf{Z}^+$ such that $(\sigma - 1) < n$. Therefore, we have $\sigma < (n+1)$. But $(n + 1) \in \mathbf{Z}^+$, so by definition of σ, we have $(n + 1) \leq \sigma$. This yields a contradiction. We therefore conclude that $\mathbf{Z}^+$ is not bounded above. Thus $\mathbf{Z}^+$ is not bounded. □

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