Do Isomorphic Graphs Induce Homeomorphic Topological Spaces?

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Given two graphs $H = (X_1, E_1)$ and $G = (X_2, E_2)$, we say that H and G are identical in the graph theory's point of view if there exists a bijective function $f : X_1 \rightarrow X_2$ such that f preserves adjacency, i.e., $[a,b] \in E_1$ if and only if $[f(a),f(b)] \in E_2$. In this case we call f an isomorphism and say that H and G are isomorphic.

Recall that an arbitrary finite graph $G^* = (X,E)$ induces a topology T_{G^*} on X with a base consisting of the sets $F(A) = X \setminus N(A)$ where

$$N(A) = A \cup \{ x : [x,a] \in E \text{ for some } a \in A \}$$

and A ranges over all subsets (finite) of X (see [1] and [2]). From this fact, two natural interesting questions arise.

Question 1. Do isomorphic graphs induce homeomorphic topological spaces?

Question 2. Can one find two non-isomorphic graphs that induce homeomorphic topological spaces?

In this short note, we shall give positive answers to both questions. Thus, while isomorphic graphs induce homeomorphic topological spaces, the converse is not necessarily true.

We need to recall some basic definitions.

Definition 1. A mapping $f : (X,\mathbf{T}_1) \rightarrow (Y,\mathbf{T}_2)$ is said to be ***continuous*** if and only if $f^{-1}(O) \in \mathbf{T}_1$ for every $O \in \mathbf{T}_2$.

Definition 2. A mapping $f : (X,\mathbf{T}_1) \rightarrow (Y,\mathbf{T}_2)$ is said to be ***open*** if and only if $f(V) \in \mathbf{T}_2$ for every $V \in \mathbf{T}_1$.

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Definition 3. Two spaces $(X,\mathbf{T}_1)$ and $(Y,\mathbf{T}_2)$ are ***homeomorphic*** if there exists a bijective (one-one and onto) and bicontinuous (open and continuous) function $f:(X,\mathbf{T}_1)\to(Y,\mathbf{T}_2)$. The function f is called a ***homeomorphism***.

We need the following lemma.

Lemma 4. Let $H=(X_1,E_1)$ and $G=(X_2,E_2)$ be isomorphic graphs and $f:X_1\to X_2$ an isomorphism. Then

(i) $f(F(A))=F(f(A))$ for every $A\subseteq X_1$, and

(ii) $f^{-1}(F(B))=F(f^{-1}(B))$ for every $B\subseteq X_2$.

Proof. (i) Let $A\subseteq X_1$ and $y\in f(F(A))$. Then there exists $x\in F(A)$ such that $f(x)=y$. Hence, by definition, $x\notin A$ and $[x,a]\notin E_1$ for all $a\in A$. Thus, $y\notin f(A)$ and $[y,z]\notin E_2$ for all $z\in f(A)$. This shows that $y\in F(f(A))$.

Conversely, let $y\in F(f(A))$. Then $y\notin f(A)$ and $[y,t]\notin E_2$ for all $t\in f(A)$. Since f is onto there exists $x\in X_1$ such that $f(x)=y$. Also, since $y\notin f(A)$, it follows that $x\notin A$. Now, suppose that $[x,a]\in E_1$ for some $a\in A$. Then $[f(x),f(a)]=[y,f(a)]\in E_2$ where $f(a)\in f(A)$. This is a contradiction. Thus, $[x,a]\notin E_1$ for all $a\in A$. This shows that $x\in F(A)$ and hence, $y\in f(F(A))$.

(ii) Let $B\subseteq X_2$ and $x\in f^{-1}(F(B))$. Then $f(x)\in F(B)$. This means that $f(x)\notin B$ and $[f(x),y]\notin E_2$ for all $y\in B$. Since $f(x)\notin B$, $x\notin f^{-1}(B)$. Also, $[x,z]\notin E_1$ for all $z\in f^{-1}(B)$. For if $[x,z]\in E_1$ for some $z\in f^{-1}(B)$, then $[f(x),f(z)]\in E_2$ where $f(z)\in B$, a contradiction. Hence $x\in f^{-1}(F(B))$.

Next, let $x\in F(f^{-1}(B))$. Then $x\notin f^{-1}(B)$ and $[x,t]\notin E_1$ for all $t\in f^{-1}(B)$. Hence, $f(x)\notin B$. Suppose there exists $u\in B$ such that $[f(x),u]\in E_2$. Let $z\in X_1$ with $f(z)=u$. Then $z\in f^{-1}(B)$ and $[f(x),f(z)]\in E_2$. Hence, $[x,z]\in E_1$, a contradiction. Thus, $[f(x),u]\notin E_2$ for all $u\in B$. Therefore, $f(x)\in F(B)$ and hence, $x\in f^{-1}(F(B))$. Accordingly, (i) and (ii) hold. □

The next result answers the Question 1.

Theorem 5. If $H=(X_1,E_1)$ and $G=(X_2,E_2)$ are isomorphic, then the induced topological spaces $(X_1,\mathbf{T}_H)$ and $(X_2,\mathbf{T}_G)$ are homeomorphic.

Proof. Let $f: X_1 \to X_2$ be an isomorphism. Then by definition, f is bijective. Hence, it remains to show that f is bicontinuous (with respect to $\mathbf{T}_H$ and $\mathbf{T}_G$).

To this end, let $O \in \mathrm{T}_H$ and $y \in f(O)$. Then there exists $x \in O$ such that $f(x) = y$. Also, there is $A \subseteq X_1$ such that $x \in F(A) \subseteq O$ since
$\{F(A) : A \subseteq X_1\}$ is a base for $\mathbf{T}_H$. Clearly, $y \in f(F(A)) \subseteq f(O)$. By Lemma 4(i), $f(F(A)) \in \mathbf{T}_G$ since $F(f(A)) \in \{F(B) : B \subseteq X_2\}$ which is a base for T_G. Thus, $f(O) \in \mathrm{T}_G$ and hence, f is open.

Now, let $V \in \mathbf{T}_G$ and $x \in f^{-1}(V)$. Then $f(x) \in V$. Hence, there exists $B \subseteq X_2$ such that $f(x) \in F(B) \subseteq V$. Thus $x \in f^{-1}(F(B)) \subseteq f^{-1}(V)$. By Lemma 4(ii), $f^{-1}(F(B))$ is a basic element of $\mathbf{T}_H$. Hence $f^{-1}(V)$ is an open subset of X_1. Therefore, f is continuous.

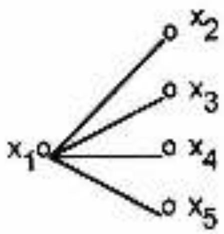
Combining the above results, we then obtain the desired result. □

Example 6. Let K_1 be the graph consisting of a single vertex, say v, and G any graph. Then $(X_1, \mathbf{T}_H)$ and $(X_2, \mathbf{T}_G)$ are homeomorphic, where $H = K_1[G]$, the composition of K_1 and G. For a thorough discussion of the graph H, see [4].

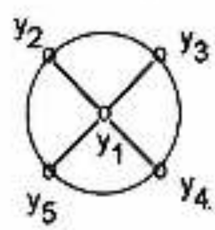
Proof. It is easy to show that the function $f: X_2 \to X_1$, defined by $f(x) = (v,x)$ is an isomorphism of the graphs G and H. Thus, by Theorem 5, the induced topological spaces are homeomorphic. □

We shall now answer the Question 2 by constructing two non-isomorphic graphs H and G which induce homeomorphic topological spaces.

Consider the star graph $H = K_{1,4}$ and the wheel graph $G = W_4$ given below:



$H = K_{1,4}$



$G = W_4$

Then $F(\{x_1\}) = \varnothing$, $F(\{x_3, x_4, x_5\}) = \{x_2\}$, $F(\{x_2, x_4, x_5\}) = \{x_3\}$, $F(\{x_2, x_3, x_5\}) = \{x_4\}$, $F(\{x_2, x_3, x_4\}) = \{x_5\}$. Observe that there exists no subset A of the vertex set of H with $F(A) = \{x_1\}$. Therefore, if we set $X_1 = \{x_1, x_2, x_3, x_4, x_5\}$, then

$$\mathbf{T}_H = \{X_1\} \cup \{ A \subseteq X_1 : x_1 \notin A \}.$$

Similarly, if $X_2 = \{y_1, y_2, y_3, y_4, y_5\}$, then

$$\mathbf{T}_G = \{X_2\} \cup \{ B \subseteq X_2 : y_1 \notin B \}.$$

Define $f : (X_1, \mathbf{T}_H) \to (X_2, \mathbf{T}_G)$ by $f(x_i) = y_i$ for $i = 1, 2, 3, 4, 5$. Then f is bijective. Now, let $O \in \mathbf{T}_H$. Then $O = X_1$ or $x_1 \notin O$. Hence, $f(O) = X_2$ or $y_1 \notin f(O)$. Thus, $f(O) \in \mathbf{T}_G$ and hence, f is open. On the other hand, if $V \in \mathbf{T}_G$, then $V = X_2$ or $y_1 \notin V$. Hence $f^{-1}(V) = X_1$ or $x_1 \notin f^{-1}(V)$. This shows that $f^{-1}(V) \in \mathbf{T}_H$. Thus, f is continuous. Therefore, f is a homeomor-phism and the two spaces are homeomorphic. However, it is obvious that H and G are not isomorphic.

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