A Theorem on the Box Topology Revisited

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Drobot and Sawka [2] provided the following characterization of continuous functions from a given metric space into a product of metric spaces with the box topology.

Theorem 1. Let X and X_α $(\alpha \in \Lambda)$ be metric spaces. A function $f: X \to \Pi_\alpha X_\alpha$ is continuous in the box topology if and only if

(i) each coordinate function $f_\alpha = \pi_\alpha \circ f: X \to X_\alpha$ is continuous, and

(ii) each point $x \in X$ has neighborhood on which all but a finite number of the f_α's are constant.

The following result was also obtained.

Corollary. Let X and X_α $(\alpha \in \Lambda)$ be metric spaces with X compact. Then $f: X \to \Pi_\alpha X_\alpha$ is continuous in the box topology if and only if

(i) all the f_α's are continuous, and

(ii) only a finite number of them are not constant.

In [1], Carpio showed the falsity of the above corollary by constructing an example of a continuous function f from a compact metric space into a product space with the box topology that fails to satisfy conditon (ii). Hence, in order for the above corollary to hold, the space X must be something more than just being compact. Carpio remarked farther that the defect can easily be remedied if we assume that X is also connected. For completeness, we will state and prove the correct version of the corollary.

Corollary 2. Let X and X_α $(\alpha \in \Lambda)$ be metric spaces with X compact and connected. Then $f: X \to \Pi_\alpha X_\alpha$ is continuous if and only if

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(i) all the f_α's are continuous, and

(ii) only a finite number of them are not constant.

Proof. ($\Leftarrow$) This is immediate because the restriction of a constant function to any subset of its domain is also constant, i.e., (ii) of Corollary 2 implies (ii) of Theorem 1.

($\Rightarrow$) Suppose that f is continuous. Then, by Theorem 1, each coordinate function f_α is continuous and for each $x \in X$ there exists an open neighborhood U_x of x such that only a finite number of the f_α's are not constant. Clearly, the class $\mathfrak{I} = \{U_x : x \in X\}$ is an open cover for X. Since X is compact, there exist $x(1), x(2), x(3), x(4), \ldots, x(n)$ in X such that

$$X = \bigcup_{i=1}^{n}(U_{x(i)})$$

Consider the class $\vartheta = \{f_\alpha : f_\alpha$ is not constant on $U_{x(i)}$ for some i, $1 \leq i \leq n\}$. Then ϑ is a finite set. Now, if $f_\beta \notin \vartheta$, then f_β is constant on $U_{x(i)}$ for all i. Let k be a constant such that $f_\beta(t) = k$ on $U_{x(1)}$. We have the following

Claim. $f_\beta(t) = k$ on $U_{x(i)}$ for all $i = 1, 2, \ldots, n$.

To prove our claim, it suffices to show that the class

$$\wp = \{ U_{x(i)} : f_\beta(t) \neq k \text{ on } U_{x(i)}\}$$

is empty. So, suppose that $\wp$ is nonempty. Let A be the union of the sets $U_{x(i)}$ in $\wp$ and B the union of those which are not in $\wp$. Then, by assumption, A is nonempty. Since $U_{x(1)}$ is contained in B, B is also nonempty. Moreover, A and B are disjoint open sets such that $A \cup B = X$. Hence, $A \cup B$ forms a disconnection of X. This is a contradiction to the assumption that X is connected. Thus, $\wp$ is empty and hence, our claim holds.

Note that the above claim implies that f_β is constant on X. Accordingly, only a finite number of the f_α s are not constant on X. □

The above characterization is important because it points out the distinction between a continuous function in the Tychonoff topology and the one in the box topology. Apparently, every continuous function into the product set with the box topology is continuous with respect to the Tychonoff topology, but not conversely. This easily follows from the fact that the Tychonoff topology on the product set is strictly coarser (or smaller) than the box topology.

In what follows, we shall show a couple of simple applications of Corollary 2. More precisely, we shall use it to prove some assertions on $\mathbf{R}^{\omega}$, the countable infinite copies of the real line, with the box topology. First, we need the following definitions.

Definition 3. Let I be the compact unit interval $[0,1]$. A ***path*** from a point x to a point y in a topological space $(Y,\mathbf{T})$ is a continuous function $f: I \rightarrow Y$ with $f(0) = x$ and $f(1) = y$. A space $(Y,\mathbf{T})$ is said to be ***path connected*** if for every pair of points $x, y \in Y$, there exists a path f from x to y.

Definition 4. A space $(Y,\mathbf{T})$ is said to be ***totally pathwise disconnected*** if and only if the only continuous functions from $[0,1]$ into Y are constant.

We now prove the following:

Theorem 5. $\mathbf{R}^{\omega}$ with the box topology is not path connected.

This result could actually follow from the result that the space $\mathbf{R}^{\omega}$ with the box topology is not a connected space. For a proof that the given space is not connected, see refs. [2] and [3]. We shall now present a simpler proof that uses Corollary 2.

Proof. Let $x = (a_1, a_2, \ldots)$ and $y = (b_1, b_2, \ldots)$ be two points in $\mathbf{R}^{\omega}$ with $a_i \neq b_i$ for all i (of course such points exist). Suppose there exists a continuous function $f: I \rightarrow \mathbf{R}^{\omega}$ such that $f(0) = x$ and $f(1) = y$. Now, let f_β be any coordinate function. Then

$$f_\beta(0) = \pi_\beta(f(0)) = \pi_\beta(x) = a_\beta, \text{ and}$$

$$f_\beta(1) = \pi_\beta(f(1)) = \pi_\beta(y) = b_\beta.$$

Since $a_\beta \neq b_\beta$, f_β is not constant. Now, since f_β was arbitrarily chosen, it follows that none of the coordinate functions can be constant. This contradicts condition (ii) of Corollary 2. Thus, there exists no path joining x and y. Therefore, $\mathbf{R}^{\omega}$ with the box topology cannot be path connected. □

Finally, we have

Theorem 6. $\mathbf{R}^{\omega}$ with the box topology is not totally pathwise disconnected.

Proof. We define a function $f = (f_1, f_2, \ldots, f_n, \ldots) : I \to \mathbf{R}^\omega$ by defining the coordinate functions $f_n : I \to \mathbf{R}$ as follows:

$$f_1(t) = t \text{ for all } t \in I,$$

and for all $n \neq 1$,

$$f_n(t) = k \ (k \in \mathbf{R}) \text{ for all } t \in I.$$

Then by Corollary 2, f is continuous with respect to the box topology. Since

$$f(0) = (0, k, k, \ldots, k, \ldots) \neq (1, k, k, \ldots, k, \ldots) = f(1),$$

it follows that f is not a constant function. Therefore, $\mathbf{R}^\omega$ with the box topology is not totally pathwise disconnected. □

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