Bilinear Henstock - Stieltjes Integral in Banach Spaces

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Francisco and Chew [FC] defined the Henstock-Stieltjes integral of a real-valued function with respect to a function of bounded variation on the compact interval $[a,b]$. This integral turned out to be equivalent to the Perron-Stieltjes integral [W].

In this paper, we shall define a more generalized concept of the Henstock-Stieltjes integral. We shall investigate its simple properties and formulate some convergence theorems.

Throughout this paper, all functions considered are bounded and are defined on the closed interval $[a,b]$. The letters X, Y, and Z are used to denote Banach spaces over the field $\mathbf{R}$ of real numbers.

Definition 1. A transformation $A : X \times Y \to Z$ is said to be **bilinear** if it satisfies the following properties:

(i) $A(x_1 + x_2, y) = A(x_1, y) + A(x_2, y)$

(ii) $A(x, y_1 + y_2) = A(x, y_1) + A(x, y_2)$

(iii) $A(\alpha x, y) = \alpha A(x, y)$

(iv) $A(x, \alpha y) = \alpha A(x, y)$

for all $x_1, x_2, x \in X$, $y_1, y_2, y \in Y$, and $\alpha \in \mathbf{R}$.

Definition 2. A bilinear transformation $A : X \times Y \to Z$ is said to be **bounded** if there exists a positive constant M such that

$$\|A(x,y)\|_Z \leq M\|x\|_X\|y\|_Y$$

for all $x \in X$ and for all $y \in Y$. Furthermore, we have

$$\|A\| = \inf\{M : \|A(x,y)\|_Z \leq M\|x\|_X \|y\|_Y, \text{ for all } x \in X \text{ and all } y \in Y\}.$$

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We denote by $L(X,Y;Z)$ the space of all bounded bilinear transformations $A : X \times Y \to Z$.

Definition 3. A function $f : [a,b] \to Y$ is said to be of **bounded variation** on $[a,b]$ if

$$\mathbf{V}(f; [a,b]) := \sup \, (D)\sum \|f(v) - f(u)\|_Y$$

is finite, where the supremum is over all divisions $D = \{[u,v]\}$ of $[a,b]$.

In what follows, $f_1 : [a,b] \to X$ and $f_2 : [a,b] \to Y$.

Definition 4. Let $C = (c_1, c_2)$ be an ordered system, where $c_j \in \{0,1\}$ for $j = 1, 2$ and let $A \in L(X,Y;Z)$. Further, we write

$$d_j f_j = f_j \text{ if } c_j = 0 \text{ and } d_j f_j = df_j \text{ if } c_j = 1.$$

We say that the **Henstock-Stieltjes**, or simply **HS**, **integral exists** if there is a vector $J \in Z$ satisfying the following property: For every $\varepsilon > 0$, there exists a positive function δ on $[a,b]$ such that for every δ-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$\|(D)\sum A(f_1(u,v), f_2(u,v)) - J\|_Z < \varepsilon,$$

where

$$f_j(u,v) = f_j(\xi), \text{ if } c_j = 0, \text{ and } f_j(u,v) = f_j(v) - f_j(u), \text{ if } c_j = 1.$$

Recall that a division $D = \{([u,v];\xi)\}$ is **δ-fine** if

$$\xi \in [u,v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi)).$$

In Definition 4, we write

$$(HS) \int_a^b A(d_1 f_1, d_2 f_2) = J.$$

If $X = Y = \mathbf{R}$, f_2 is of bounded variation on $[a,b]$, $A(r_1, r_2) = r_1 r_2$ and $C = (0,1)$, then Definition 4 reduces to the Henstock- Stieltjes integral by Francisco and Chew in [FC]. Also, it is easy to show that if the HS integral exists, then it is unique.

The following theorem is known as the **Cauchy criterion** for the existence of the integral.

Theorem 5. Let $C = (c_1, c_2)$ and $A \in \boldsymbol{L}(X,Y;Z)$. Then

$$(\text{HS}) \int_a^b A(d_1 f_1, d_2 f_2)$$

exists if and only if for every $\varepsilon > 0$, there exists a function $\delta(\xi) > 0$ on $[a,b]$ such that for any two δ-fine divisions $D_1 = \{([u,v];\xi)\}$ and $D_2 = \{([u',v'];\xi')\}$ of the interval $[a,b]$,

$$\|(D_1)\sum A(f_1(u,v), f_2(u,v)) - (D_2)\sum A(f_1(u',v'), f_2(u',v'))\|_Z < \varepsilon.$$

Proof. ($\Rightarrow$): Suppose that the HS integral exists and is equal to $J \in Z$. Then given $\varepsilon > 0$, there exists $\delta(\xi) > 0$ such that for any δ-fine divisions D_1 and D_2 of $[a,b]$.

$$\|(D_1)\sum A(f_1(u,v), f_2(u,v)) - J\|_Z < \varepsilon/2 \text{ and}$$

$$\|(D_2)\sum A(f_1(u,v), f_2(u,v)) - J\|_Z < \varepsilon/2.$$

Thus,

$$\|(D_1)\sum A(f_1(u,v), f_2(u,v)) - (D_2)\sum A(f_1(u,v), f_2(u,v)\|_Z < \varepsilon.$$

($\Leftarrow$): For each positive integer n, there exists a $\delta_n(\xi)$ such that

$$\|(D_1)\sum A(f_1(u,v), f_2(u,v)) - (D_2)\sum A(f_1(u,v), f_2(u,v))\|_Z < 1/n$$

whenever D_1 and D_2 are δ_n-fine divisions of $[a,b]$. We may assume that $\delta_{n+1}(\xi) \leq \delta_n(\xi)$ for all n. Now, for each n, fix a δ_n-fine division $\mathbf{D}_n$ and put

$$s_n = (D_n)\sum A(f_1(u,v), f_2(u,v)).$$

Let n, m be positive integers with $n < m$. Then D_m is both a δ_m-fine and δ_n-fine division of $[a,b]$. Thus,

$$\| s_n - s_m \|_Z < 1/n$$

and hence, $\{s_n\}$ is a Cauchy sequence in Z. Since Z is complete, there exists a vector J in Z such that

$$\lim_{n\to\infty} s_n = J.$$

Thus, given $\varepsilon > 0$, there is a positive integer N such that for all $n \geq N$,

$$\| s_n - J \|_Z < \varepsilon/2.$$

Choose $N^* \geq N$ such that $1/N^* < \varepsilon/2$. Define $\delta(\xi) = \delta_{N^*}(\xi)$ for each ξ in $[a,b]$. Then for any δ-fine division D of $[a,b]$, we have

$$\| (D)\sum A(f_1(u,v), f_2(u,v)) - J \|_Z \leq \| (D)\sum A(f_1(u,v), f_2(u,v)) - s_{N^*} \|_Z +$$

$$+ \| s_{N^*} - J \|_Z < 1/N^* + \varepsilon/2 < \varepsilon.$$

Therefore, the HS integral exists and

$$\text{(HS)} \int_a^b A(d_1 f_1, d_2 f_2) = J. \quad \square$$

Theorem 6. Let $A \in L(X,Y;Z)$. If f_1 is continuous and f_2 is of bounded variation on $[a,b]$, then,

(i) (HS) $\int_a^b A(f_1, df_2)$ exists;

(ii) (HS) $\int_a^b A(df_1, df_2)$ exists and is equal to θ_Z, the zero vector.

Proof. (i) Since f_1 is continuous on $[a,b]$, it is uniformly continuous there. Hence given $\varepsilon > 0$, there exists $\eta > 0$ such that for all $t, t' \in [a,b]$,

$$|t - t'| < \eta \text{ implies } \| f_1(t) - f_1(t') \|_Z < \varepsilon.$$

Define $\delta(\xi) = \eta/2$ for all $\xi \in [a,b]$. Let D_1 and D_2 be two δ-fine divisions of

$[a,b]$. Then there exists a δ-fine division D_3 of $[a,b]$ which is finer than both D_1 and D_2. Now, let $[u,v]$ be an interval division in D_1. Then there exist division points $z_0, z_1, \ldots, z_r$ in D_3 such that $u = z_0 < z_1 < \ldots < z_r = v$. Consider the following difference:

$$\Delta(u,v) = A(f_1(\xi), f_2(v) - f_2(u)) - \sum_{k=1}^{r} A(f_1(\xi_k), f_2(z_k) - f_2(z_{k-1}))$$

$$= \sum_{k=1}^{r} A(f_1(\xi) - f_1(\xi_k), f_2(z_k) - f_2(z_{k-1})).$$

Then

$$\|\Delta(u,v)\|_Z \leq \sum_{k=1}^{r} \|A\| \, \| f_1(\xi) - f_1(\xi_k) \|_X \, \| f_2(z_k) - f_2(z_{k-1}) \|_Y$$

$$< \varepsilon \|A\| \mathbf{V}(f_2;[u,v]).$$

It follows that

$$\| (D_1)\sum A(f_1(\xi), f_2(v) - f_2(u)) - (D_3)\sum A(f_1(\xi), f_2(v) - f_2(u)) \|_Z \leq$$

$$\leq (D_1)\sum \| \Delta(u,v) \|_Z < \varepsilon \|A\| \mathbf{V}(f_2; [a,b]).$$

Similarly,

$$\| (D_2)\sum A(f_1(\xi), f_2(v) - f_2(u)) - (D_3)\sum A(f_1(\xi), f_2(v) - f_2(u)) \|_Z <$$

$$< \varepsilon \|A\| \mathbf{V}(f_2; [a,b]).$$

Therefore,

$$\| (D_1)\sum A(f_1(\xi), f_2(v) - f_2(u)) - (D_2)\sum A(f_1(\xi), f_2(v) - f_2(u)) \|_Z <$$

$$< 2\varepsilon \|A\| \mathbf{V}(f_2;[a,b]).$$

By the Cauchy criterion (Theorem 5), we have the desired result.

(ii) As in (i), define $\delta(\xi) = \eta/2$. Let $D = \{([u,v],\xi)\}$ be a δ-fine division of $[a,b]$. Then

$$\|(D)\sum A(f_1(v) - f_1(u), f_2(v) - f_2(u))\|_Z < \varepsilon \|A\| \mathbf{V}(f_2;[a,b]).$$

This shows that

$$\text{(HS)} \int_a^b A(df_1, df_2) = \theta_Z. \quad \square$$

Simple Properties

We shall show that the HS integral has the usual properties of integrals.

Theorem 7. Let $A,\ B \in \boldsymbol{L}(X,Y,Z),\ f_1,\ f : [a,b] \to X$ and $g_1, g : [a,b] \to Y$.

(i) If (HS) $\int_a^b A(f,dg)$ exists, then (HS) $\int_c^d A(f,dg)$ exists for every subinterval $[c,d]$ of $[a,b]$.

(ii) If $\lambda \in \mathbf{R}$ and the integral (HS) $\int_a^b A(f,dg)$ exists, then the integrals (HS) $\int_a^b A(\lambda f,dg)$ and (HS) $\int_a^b A(f,d[\lambda g])$ exist. Moreover,

$$\text{(HS)} \int_c^b A(\lambda f,dg) = \lambda\ \text{(HS)} \int_a^b A(f,dg) \text{ and}$$

$$\text{(HS)} \int_a^b A(f,d[\lambda g]) = \lambda\ \text{(HS)} \int_a^b A\,(f,dg).$$

(iii) If the integrals (HS) $\int_a^b A(f,dg)$ and (HS) $\int_a^b A(f_1,dg)$ exist, then the integral (HS) $\int_a^b A(f+f_1,\mathrm{dg})$ also exists and

$$\text{(HS)} \int_a^b A(f+f_1,dg) = \text{(HS)} \int_a^b A(f,dg) + \text{(HS)} \int_a^b A(f_1,dg).$$

(iv) If the integrals (HS) $\int_a^b A(f,dg)$ and (HS) $\int_a^b A(f,dg_1)$ exist, then the integral (HS) $\int_a^b A(f,d[g+g_1])$ also exists and

$$\text{(HS)} \int_a^b A(f,d[g+g_1]) = \text{(HS)} \int_a^b A(f,dg) + \text{(HS)} \int_a^b A(f,dg_1).$$

Proof. (i) By the Cauchy criterion, there exists $\delta_1(\xi) > 0$ on $[a,b]$ such that for any δ_1-fine divisions D and D′ of $[a,b]$, we have

$$\| (D)\sum A(f(\xi), g(u,v)) - (D')\sum A(f(\xi'),g(u',v')) \|_Z < \varepsilon.$$

Let $\delta(\xi) = \delta_1(\xi)$ for all $\xi \in [c,d]$. Let D_1 and D_2 be any δ-fine divisions of $[c,d]$. Let E_1 and E_2 be fixed δ_1-fine divisions of $[a,c]$ and $[d,b]$, respectively. Consider $D = E_1 \cup D_1 \cup E_2$ and $D' = E_1 \cup D_2 \cup E_2$. Then D and D' are δ_1-fine divisions of $[a,b]$ and

$$(D)\sum A(f(\xi),g(u,v)) - (D')\sum A(f(\xi'),g(u',v')) =$$

$$= (D_1)\sum A(f(\xi),g(u,v)) - (D_2)\sum A(f(\xi'),g(u',v')).$$

Thus,

$$\|(D_1)\sum A(f(\xi),g(u,v)) - (D_2)\sum A(f(\xi'),g(u',v'))\|_Z < \varepsilon.$$

By the Cauchy criterion applied to $[c,d]$, it follows that ,the integral

$$(HS)\int_c^d A(f,dg) \text{ exists.}$$

(ii) Let $\varepsilon > 0$ and $\lambda \in \mathbf{R}$. Since $\int_a^b A(f,dg) = J$ exists, there exists $\delta_1(\xi) > 0$ such that for any δ_1-fine division $D_1 = \{([u,v];\xi)\}$ of $[a,b]$,

$$\|(D_1)\sum A(f(\xi),g(v) - g(u)) - J\|_Z < \varepsilon.$$

Let $\delta(\xi) = \delta_1(\xi)$ for all $\xi \in [a,b]$. Then for any δ-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$,

$$\|(D)\sum A(\lambda f(\xi),g(v) - g(u)) - \lambda J\|_Z \le$$
$$\le |\lambda| \; \|(D)\sum A(f(\xi),g(v) - g(u)) - J\|_Z < |\lambda|\varepsilon, \text{ and}$$

$$\|(D)\sum A(f(\xi), \lambda(g(v) - g(u)) - \lambda J\|_Z \le$$
$$\le |\lambda| \; \|(D)\sum A(f(\xi),g(v) - g(u)) - J\|_Z < |\lambda|\varepsilon.$$

Therefore,

$$(\mathrm{HS})\int_a^b A(\lambda f, dg) = \lambda(\mathrm{HS})\int_a^b A(f, dg) \quad \text{and}$$

$$(\mathrm{HS})\int_a^b A(f, \lambda dg) = \lambda(\mathrm{HS})\int_a^b A(f, dg)$$

(iii) Let $\varepsilon > 0$. Let J and J_1 be the respective integrals. Then there exists a common $\delta(\xi) > 0$ such that for any δ-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$,

$$\|(D)\sum A(f(\xi), g(v) - g(u)) - J\|_Z < \varepsilon/2 \text{ and}$$

$$\|(D)\sum A(f_1(\xi), g(v) - g(u)) - J_1\|_Z < \varepsilon/2.$$

Thus if $D = \{([u,v];\xi)\}$ is a δ-fine division of $[a,b]$, then

$$\|(D)\sum A(f(\xi)+f_1(\xi), g(v) - g(u)) - (J + J_1)\|_Z \leq$$

$$\leq \|(D)\sum A(f(\xi), g(v) - g(u)) - J\|_Z +$$

$$+ \|(D)\sum A(f_1(\xi), g(v) - g(u)) - J_1\|_Z$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore (iii) holds.

(iv) This is proved as in (iii) (since A is also linear in the second component).

Convergence Theorems

We shall now turn our attention to the convergence theorems.

Theorem 8. Let $\{f_n : [a,b] \to X\}$ be a sequence of bounded functions which converges uniformly to the function $f : [a,b] \to X$ and let $g : [a,b] \to Y$ be a function bounded variation on $[a,b]$. If $A \in \boldsymbol{L}(X,Y;Z)$ and the integrals

$$J_n = (\mathrm{HS})\int_a^b A(f_n, dg) \text{ and } J = (\mathrm{HS})\int_a^b A(f, dg)$$

exist for all n, then

$$\lim_{n\to\infty} J_n = J.$$

Proof.: Let $\varepsilon > 0$. Then there exists a natural number N such that for all $n \geq N$ and for all $t \in [a,b]$, we have $\| f_n(t) - f(t) \|_X < \varepsilon$. Let $n \geq N$ be fixed. Since J_n and J exist, there exists a common $\delta(\xi) > 0$ on $[a,b]$ such that for any δ-fine division $D = \{([u,v],\xi)\}$ of $[a,b]$, we have

$$\| J_n - (D)\sum A(f_n(\xi), g(v) - g(u)) \|_Z < \varepsilon \text{ and}$$

$$\| J - (D)\sum A(f(\xi), g(v) - g(u)) \|_Z < \varepsilon.$$

It follows that

$$\| J_n - J \| < 2\varepsilon + \varepsilon \| A \| V(g,[a,b]).$$

Accordingly,

$$\lim_{n\to\infty} J_n = J. \quad \square$$

Definition 9. Let $A \in L(X,Y;Z)$ and $g : [a,b] \to Y$ be a function of bounded variation. We say that $f : [a,b] \to X$ is **A-integrable with respect to g on $[a,b]$** if

$$(\mathrm{HS})\int_a^b A(f, dg) \text{ exists.}$$

From this point on , all integral are, unless otherwise specefied, HS integrals.

Definition 10. Let $g : [a,b] \to Y$ be a function of bounded variation and $A \in L(X,Y;Z)$. Let $\{f_n : [a,b] \to X\}$ be a sequence of A-integrable functions with respect to g on $[a,b]$. We say that $\{f_n\}$ is **$\gamma(A)$–convergent** to $f : [a,b] \to X$ with respect to g if for every $\varepsilon > 0$, there exists a natural number N_ε such that if $k \geq N_\varepsilon$ there exists $\gamma_k(\xi) > 0$ defined on $[a,b]$ such that for any γ_k-fine division $D = \{([u,v],\xi)\}$ of $[a,b]$,

$$\|(D)\sum A(f_k(\xi) - f(\xi), g(v) - g(u))\|_Z < \varepsilon.$$

Example 11. Let $A \in \boldsymbol{L}(X,Y;Z)$ and $g : [a,b] \to X$ be a function of bounded variation on $[a,b]$. If $\{f_n\}$ is a sequence of X-valued continuous functions that converges uniformly to $f : [a,b] \to X$, then $\{f_n\}$ is $\gamma(A)$-convergent to f on $[a,b]$ with respect to g.

Proof. First, we note that in view of Theorem 6(i) the integral $\int_a^b A(f_n, dg)$ exists for each n. Now, let $\varepsilon > 0$. Since $f_n \to f$ uniformly on $[a,b]$ as $n \to \infty$, there exists a natural number $N(\varepsilon) = N$ such that if $k \geq N$ and $t \in [a,b]$, then

$$\|f_k(t) - f(t)\|_X < \varepsilon.$$

Let $D = \{([u,v];\xi)\}$ be any interval-point pair division of $[a,b]$. Then

$$\|(D)\sum A(f_k(\xi)-f(\xi),g(v)-g(u))\|_Z \leq \|A\|(D)\sum \|f_k(\xi)-f(\xi)\| \, \|g(v)-g(u)\|_Z$$

$$< \varepsilon\|A\|\mathbf{V}(g;[a,b]).$$

Therefore $\{f_n\}$ is $\gamma(A)$-convergent to f with respect to f. □

Theorem 12. Let $A \in \boldsymbol{L}(X,Y;Z)$ and $g : [a,b] \to Y$ a function of bounded variation. Then the sequence $\{f_n : [a,b] \to X\}$ of A-integrable functions with respect to g is $\gamma(A)$-convergent to $f : [a,b] \to X$ with respect to g if and only if

$$\int_a^b A(f,dg) \text{ exists and } \lim_{n\to\infty} \int_a^b A(f_n,dg) = \int_a^b A(f,dg).$$

Proof. ($\Rightarrow$) Let $\varepsilon > 0$ and let $N(\varepsilon) = N$ be as in Definition 10. If h, $k \geq N$, then there exists a $\gamma(\xi) > 0$ (depending on h and k) such that if $D_1 = \{([u,v];\xi)\}$ is a γ-fine division of $[a,b]$, then

$$\|(D_1)\sum A(f_k(\xi), g(v) - g(u)) - (D_1)\sum A(f(\xi), g(v) - g(u))\|_Z < \varepsilon \text{ and}$$

$$\|(D_1)\sum A(f_h(\xi), g(v) - g(u)) - (D_1)\sum A(f(\xi), g(v) - g(u))\|_Z < \varepsilon.$$

Since f_k and f_h are A-integrable with respect to g on $[a,b]$, there exists $\delta(\xi) > 0$ such that if $\mathbf{D}_2 = \{([u,v];\xi)\}$ is any δ-fine division of $[a,b]$,

$$\|(D_2)\sum A(f_k(\xi), g(v) - g(u)) - \int_a^b A(f_k, dg)\|_Z < \varepsilon \text{ and}$$

$$\|(D_2)\sum A(f_h(\xi), g(v) - g(u)) - \int_a^b A(f_h, dg)\|_Z < \varepsilon.$$

Put $\eta(\xi) = \min\{\gamma(\xi), \delta(\xi)\}$ for every $\xi \in [a,b]$. Therefore if $D = \{([u,v];\xi)\}$ is an η-fine division of $[a,b]$, then

$$\begin{aligned}
&\|\int_a^b A(f_k, dg) - \int_a^b A(f_h, dg)\|_Z \le \\
&\le \left\|\int_a^b A(f_k, dg) - (D)\sum A(f_k(\xi), g(v) - g(u))\right\|_Z + \\
&\quad + \|(D)\sum A(f_k(\xi), g(v) - g(u)) - (D)\sum A(f(\xi), g(v) - g(u))\|_Z + \\
&\quad + \|(D)\sum A(f(\xi), g(v) - g(u)) - (D)\sum A(f_h(\xi), g(v) - g(u))\|_Z + \\
&\quad + \|(D)\sum A(f_h(\xi), g(v) - g(u)) - \int_a^b A(f_h, dg)\|_Z \\
&< \varepsilon + \varepsilon + \varepsilon + \varepsilon = 4\varepsilon.
\end{aligned}$$

Since ε is arbitrary, it follows that the sequence $\left\{\int_a^b A(f_n, dg)\right\}$ is a Cauchy sequence in Z. Hence there exists $J \in Z$ such that

$$J = \lim_{n\to\infty} \int_a^b A(f_n, dg).$$

Claim: J is the A-integral of f with respect to g on $[a,b]$. Let $\varepsilon > 0$. Then there exists a natural number $N^*(\varepsilon) = N^*$ such that for all $k \geq N^*$,

$$\left\| \int_a^b A(f_k, dg) - J \right\|_Z < \varepsilon.$$

Set $M = M(\varepsilon) = \max\{N, N^*\}$, where $N = N(\varepsilon)$ is as in Definition 10. Then for $k \geq M$ there exists $\gamma_k(\xi) > 0$ such that if $D^* = \{([u,v];\xi)\}$ is a γ_k-fine division of $[a,b]$, then

$$\left\|(D^*)\sum A(f_k(\xi) - f(\xi), g(v) - g(u)\right\|_Z < \varepsilon.$$

Also, there exists $\delta_k(\xi) > 0$ such that if $D^{**} = \{([u,v];\xi)\}$ is a δ_k-fine division of $[a,b]$, then

$$\left\|(D^{**})\sum A(f_k(\xi), g(v) - g(u)) - \int_a^b A(f_k, dg)\right\| < \varepsilon.$$

Define $\eta_k(\xi) = \min\{\gamma_k(\xi), \delta_k(\xi)\}$ for every $\xi \in [a,b]$. Therefore if $D = \{([u,v];\xi)\}$ is a η_k-fine division of $[a,b]$, then

$$\begin{aligned} &\|(D)\sum A(f(\xi), g(v) - g(u)) - J\| \leq \\ &\leq \|(D)\sum A(f(\xi), g(v) - g(u)) - D)\sum A(f_k(\xi), g(v) - g(u))\| \\ &\quad + \|(D)\sum A(f_k(\xi), g(v) - g(u)) - \int_a^b A(f_k, dg)\| + \\ &\quad + \|\int_a^b A(f_k, dg) - J\| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Therefore f is A-integrable with respect to g on $[a,b]$.

($\Leftarrow$): Let $\varepsilon > 0$. Then there exists a natural number $N = N(\varepsilon)$ such that if $k, h \geq N$ then

$$\left\| \int_a^b A(f_k, dg) - \int_a^b A(f_h, dg) \right\| < \varepsilon.$$

Let $k \geq N$ be fixed (but arbitrary). Since f_k and f are A-integrable with respect to g on $[a,b]$, ther exists a common $\delta_k(\xi) > 0$ on $[a,b]$ such that if $D^* = \{([u,v];\xi)\}$ is a δ_k-fine division of $[a,b]$, then

$$\|(D^*)\sum A(f_k(\xi), g(v) - g(u)) - \int_a^b A(f_k, dg)\| < \varepsilon \text{ and}$$

$$\|(D^*)\sum A(f(\xi), g(v) - g(u)) - \int_a^b A(f, dg)\| < \varepsilon.$$

Define $\gamma_k(\xi) = \delta_k(\xi)$ for $\xi \in [a,b]$. Then for any γ_k-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$,

$$\|(D)\sum A(f_k(\xi) - f(\xi), g(v) - g(u))\| \leq$$

$$\leq \|(D)\sum A(f_k(\xi), g(v) - g(u)) - \int_a^b A(f_k, dg)\| +$$

$$+ \|\int_a^b A(f_k, dg) - \int_a^b A(f, dg)\| +$$

$$+ \|\int_a^b A(f, dg) - (D)\sum A(f(\xi), g(v) - g(u)\| < 3\varepsilon.$$

Therefore $\{f_n\}$ is $\gamma(A)$-convergent to f with respect to g. □

We now use the above theorem to prove our last result.

Theorem 13. Let $C([a,b],X)$ be the space of X-valued continuous functions on $[a,b]$ with the uniform norm. Let $g : [a,b] \to Y$ be function of bounded variation on $[a,b]$ and $A \in L(X,Y;Z)$. Then

$$T(f) = \int_a^b A(f,dg), \ f \in C([a,b],X),$$

defines a continuous linear operator on $C([a,b],X)$ into Z.

Proof. First, note that the above integral exists in view of Theorem 6. Now linearity of T follows from Theorem 7(ii) and (iii). It remains to show that T is continuous.

To this end, let $f, f_n \in C([a,b],X)$ for $n = 1, 2, \ldots$, and

$$\|f_n - f\|_\infty \to 0 \text{ as } n \to \infty.$$

By Example 11, the sequence $\{f_n\}$ is $\gamma(A)$-convergent to f with respect to g. Thus, by Theorem 12, we have

$$\lim_{n\to\infty} \int_a^b A(f_n, dg) = \int_a^b A(f, dg).$$

It follows that $\|T(f_n) - T(f)\| \to 0$ as $n \to \infty$. This shows that T is continuous on $C([a,b],X)$. □

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