Translates & Partitioning

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Let G be a group. If H is a subgroup of G, then it is known that the left cosets of H in G partition G. Suppose $A \subset G$, but A is not a subgroup of G. Do the "left cosets" of A partition G ? This paper seeks to answer this question.

Definition 1. Let $\mathfrak{I} = \{B_i\}_{i \in I}$ be a family of nonempty subsets of G. Then $\mathfrak{I}$ is a **partition** of G if

(1) $\cup_{i \in I} B_i = G$ and

(2) for any B_i and Bj, either $B_i = Bj$ or $B_i \cap B_j = \varnothing$.

Definition 2. Let A be a subset of a group G. If $g \in G$, then the **left translate** gA of A in G is the set

$$gA = \{ ga : a \in A \}.$$

Let H be a subgroup of a group G. We shall study the left translates of a coset of H in G. Let $A = gH$ for some $g \in G$. Then

$$xA = x(gH) = (xg)H.$$

This means that every left translate of A is a left coset of H. Furthermore, no two distinct left translates of A are equal to the same left coset of H.

On the other hand, for $z \in G$,

$$zH = zg^{-1}gH = zg^{-1}A.$$

This implies that every left coset of H is a left translate of A. Also, no two distinct left cosets of H are equal to the same left translate of A.

Combining the results above, we see that the left translates of A are the left cosets of H, and conversely. Since the left cosets of H partition G, we have Lemma 1.

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Lemma 1. Let G be a group and H a subgroup of G. If A is a left translate of H, then the left translates of A partition G.

Next, we prove the converse of Lemma 1. Let G be a group and $A \subset G$. If H is a left translate of A, then the left translates of H coincide with the left translates of A. To verify this, let $H = xA$ for some $x \in G$ and define a function h by $h(gH) = gx^{-1}H$. It easy to show that h is bijective. If we assume further that the left translates of A partition G, then we have the following result.

Proposition. The left translate of A containing the identity element of G is a subgroup of G.

Proof. Let H be the left translate of A containing the identity element $e \in G$. Clearly, $H \neq \varnothing$. To verify closure in H, let $a, b \in H$. Then $ab \in aH$ and $a = ae \in aH$. Hence $a \in H \cap aH$, i.e., H and aH are not disjoint. This implies that $H = aH$. Therefore, $ab \in H$. To check the existence of inverses in H, let $a \in H$. Then $a^{-1}a \in a^{-1}H$. Since $a^{-1}a = e \in H$, we have $a^{-1}a \in H \cap a^{-1}H$, i.e., $H \cap a^{-1}H \neq \varnothing$. It follows that $H = a^{-1}H$. Therefore, $a^{-1} = a^{-1}e \in a^{-1}H = H$. □

Lemma 2. Let G be a group and $A \subset G$. If the left translates of A partition G, then A is a left coset of a subgroup of G.

Proof. Let H be the left translate of A containing the identity element $e \in G$. Then $H = xA$ for some $x \in G$. This implies that $A = x^{-1}H$. This shows that A is a left coset of the subgroup H. □

Theorem. Let G be a group and $A \subset G$. Then the left translates of A partition G if and only if A is a left coset of a subgroup of G.

Proof. This theorem follows by combining Lemmas 1 and 2. □

Partitioning property arises naturally from nonempty images of a function defined on a group. First, we recall the following definition.

Definition 3. Let $f: M \to N$ be a function. The **inverse image** of $n \in N$ under f, denoted by $f^{-1}(n)$, is the set $f^{-1}(n) = \{ m \in M : f(m) = n \}$.

Let G be a group of permutations acting on a set S and let s be a fixed element of S. Consider the function $f: G \to S$ defined by $f(\pi) = \pi(s)$ for $\pi \in G$. Then we have the following notes.

Note 1. $f^{-1}(t) \neq \varnothing$ if and only if t is in the orbit of s.

Proof. Let $\delta \in f^{-1}(t) \neq \varnothing$. Then $f(\delta) = t$. But $f(\delta) = \delta(s)$; hence $\delta(s) = t$, and t is in the orbit of s. Conversely, let t be in the orbit of s. Then $t = \rho(s)$ for some $\rho \in G$. But $f(\rho) = \rho(s)$; Therefore we have $\rho \in f^{-1}(t)$; Hence, $f^{-1}(t) \neq \varnothing$: □

Note 2. $\delta f^{-1}(t) = f^{-1}[\delta(t)]$ for all $\delta \in G$.

Proof. Let $\delta \in G$. Since $f^{-1}(t) = \{ \pi \in G : f(\pi) = t \}$, we have

$$\delta f^{-1}(t) = \{ \delta\pi \in G : f(\pi) = \pi(s) = t \}.$$

Noting that $\delta\pi(s) = \delta[\pi(s)] = \delta(t)$, the last expression above can be rewritten as

$$\delta f^{-1}(t) = \{ \alpha \in G : \alpha(s) = \delta(t) \} = f^{-1}[\delta(t)]. \quad \square$$

Note 3. Let t be in the orbit of s. Then the nonempty inverse images under f coincide with the left translates of $f^{-1}(t)$.

Proof. Let $\delta \in G$. By Note 2, we have $\delta f^{-1}(t) = f^{-1}[\delta(t)]$. On the other hand, t is in the orbit of s means $\rho(s) = t$ for some $\rho \in G$. Then

$$f^{-1}[\delta(s)] = f^{-1}[\delta\rho^{-1}\rho(s)] = f^{-1}[\delta\rho^{-1}(t)] = \delta\rho^{-1}f^{-1}(t).$$

This ends the proof. □

Note 4. The nonempty inverse images of f partition G.

Proof. Clearly, $\cup_{a\in G} f^{-1}[a(s)] \subset G$. Let $\theta \in G$. Then $f(\theta) = \theta(s)$, i.e., $\theta \in f^{-1}[\theta(s)]$. This implies that $\theta \in \cup_{a\in G} f^{-1}[a(s)]$. Therefore, $G \subset \cup_{a\in G} f^{-1}[a(s)]$. Let $f^{-1}[\alpha(s)]$ and $f^{-1}[\beta(s)]$ be any two nonempty inverse images of f. If they are disjoint, then we are done. Suppose they are not disjoint and τ is in their intersection. Then $f(\tau) = \alpha(s)$ and $f(\tau) = \beta(s)$. This implies that $\alpha(s) = \beta(s)$. Hence $f^{-1}[\alpha(s)] = f^{-1}[\beta(s)]$. □

Note 5. If t is in the orbit of S, then the left translate of $f^{-1}(t)$ partition G.

Proof. Follows from the preceding notes. □.

Note 5 and the theorem imply that the left translates of $f^{-1}(t)$ are the left cosets of a subgroup of G. Now $f^{-1}(t)$ is itself a left translate. Thus, it is a left coset of a subgroup of G. It is our next task to find such a subgroup. To do this, we consider the subset $f^{-1}(s)$ of G containing permutations whose effect on s is identity:

$$f^{-1}(s) = \{ \pi \in G : \pi(s) = s \}.$$

Claim 1. $f^{-1}(s) = \{ \pi \in G : \pi(s) = s \}$ is a subgroup of G. This subgroup, denoted by Stab (s), is called the **stabilizer** of s.

Proof. $f^{-1}(s) \neq \varnothing$ since the identity permutation in G is in $f^{-1}(s)$. Let $\alpha, \beta \in f^{-1}(s)$. Then $\alpha\beta(s) = \alpha[\beta(s)] = \alpha(s) = s$, and so $\alpha\beta \in f^{-1}(s)$. We have proven closure in $f^{-1}(s)$. Now let $\alpha \in f^{-1}(s)$. Then $\alpha(s) = s$, and so $\alpha^{-1}[\alpha(s)] = I(s) = s$ where I is the identity permutation in G. Thus $\alpha^{-1} \in f^{-1}(s)$. This proves the existence of inverses in $f^{-1}(s)$. □

Claim 2. If t is in the orbit of s, then $f^{-1}(t)$ is a left coset of Stab (s).

Proof. Let $\delta(s) = t$ for some $\delta \in G$. Then

$$\text{Stab}\,(s) = \{ \pi \in G : \pi(s) = s \} = \{ \delta\pi \in G : \delta\pi(s) = \delta(s) \}$$
$$= \{ \tau \in G : \tau(s) = t \} = f^{-1}(t). \quad \square$$

Remarks. (1) The left translates of $f^{-1}(t)$ are left cosets of Stab (s).

(2) There exists a one-to-one correspondence between the orbit of s and the set of distinct left cosets of Stab (s).

(3) If G is finite, then the cardinality n of the orbit of s is

$$n = \frac{|G|}{|\text{Stab}\,(s)|}.$$

Proof. (1) Follows from Claim 2.

(2) Let Q denote the orbit of s and let $\mathfrak{R}$ denote the set of distinct left cosets of Stab (s). We define a mapping $h : Q \to \mathfrak{R}$ by

$$h[\delta(s)] = \delta f^{-1}(s) = f^{-1}[\delta(s)].$$

Clearly, h is well-defined. To show that it is one-to-one, let $h[\delta(s)] = h[\alpha(s)]$. Then $f^{-1}[\delta(s)] = f^{-1}[\alpha(s)]$. Suppose that $\delta(s) \neq \alpha(s)$. Then there exists $\beta \in G$ such that $f(\beta) = \delta(s)$ and $f(\beta) = \alpha(s)$. This contradicts the fact that f is a function. Hence $\delta(s) = \alpha(s)$, and h is one-to-one. To show ontoness, let $\delta f^{-1}(s)$ be a left coset of $f^{-1}(s)$. Then $\delta(s)$ is in the orbit of s, and so $h[\delta(s)] = \delta f^{-1}(s)$. Hence, h is onto.

(3) If G is finite, the preceding result implies that

$$n\,|\,\mathrm{Stab}\,(s)| = |\,G\,|. \quad \Box$$

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