A NOTE ON INTEGRAL FUNCTIONS

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Introduction

Integration can be developed from several points of view. One can introduce the integral through: area under the curve, limit of a sum, antiderivatives, and step functions. It can also be introduced abstractly as a functional (a la Daniel), a set function, or simply as a function of two real variables.

The function approach, which was first used by Lebesgue (see [1, p. 49], provides an interesting alternative to the standard approaches in calculus. It leads us quickly to the main results and is easy to apply. However, it never gained the attention it deserves. Until now, practically all calculus books do not mention it. One of the few exceptions is Lang's book [2]. The definition of an integral function adopted here is based on Lang's idea.

The purpose of this paper is to extend the investigation initiated in Lang's book. We shall establish some properties of the integral function and prove analogues of the theorems proved by Lang. This paper is part of a talk I delivered in a mathematics enrichment seminar held at the Northern Mindanao State Institute of Science and Technology in Butuan City. I would like to thank Dr. Jose T. de Luna and the referee for reading this paper critically. I also thank my friends at NORMISIST and their president, Engr. Alberto Villares, for their kind invitation and hospitality.

Preliminaries

The least upper bound or the supremum of a set S is denoted by sup S. The greatest lower bound or the infimum of a set S is denoted by inf S. We follow the notation and terminology in Ross [3] . In this paper

- f is a bounded real-valued function of a real variable
- B is a bound for f, i.e., $|f(t)| \leqslant B$, for all t in RR
- $P = \{x_i : u = x_0 < x_1 < x_2 < \ldots < x_n = v\}$ is a partition of the closed interval $[u, v]$
- $m_i = \inf\{f(t) : x_{i-1} \leqslant t \leqslant x_i\}$ for $i = 1, 2, \cdots, n$
- $M_i = \sup\{f(t) : x_{i-1} \leqslant t \leqslant x_i\}$ for $i = 1, 2, \cdots, n$
- $L(F = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$ is the lower sum corresponding to the partition P
- $U(P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$ is the upper sum corresponding to the partition P
- $\underline{\int_u^v} = \sup_P L(P)$ is the lower Riemann integral of f on $[u,v]$

- $\bar{\int}_u^v f(t)\,dt = \inf_P U(P)$ is the upper Riemann integral of f on $[u, v]$
- $\int_u^v f(t)\,dt$ is the Riemann integral of f on $[u, v]$

If P and Q are partitions of the same interval $[u, v]$, we say that Q is *finer* than P if the set Q contains P. In this case, we have $L(P) \leq L(Q) \leq U(Q) \leq U(P)$. If Q_1 is a partition of $[u, v]$ and Q_2 is a partition of $[v, w]$, then the set $Q_1 \cup Q_2$ is a partition of $[u, w]$. Moreover, $L(Q_1 \cup Q_2) = L(Q_1) + L(Q_2)$ and $U(Q_1 \cup Q_2) = U(Q_1) + U(Q_2)$.

The Integral as a Function

Definition 1 (Lang [2]). Let $I : RR^2 — RR^2$ be a real-valued function of two real variables satisfying the following properties:

(i) If $m \leq f(t) \leq M$, for $x \leq t \leq y$, then $m \leq (y - x) \leq I(x,y) \leq M(x, y)$.

(ii) For all x, y, z in RR, we have $I(x,y) + I(y,z) = I(x,z)$.

Then I is called an integral function of f.

Proposition 2. Let I be an integral function of f and let a be a fixed real number.

(i) For each x in RR, $I(x,x) = 0$

(ii) For all x, y in RR, $I(x,y) = I(y - x)$.

(iii) If I_0 and I_1 are integral functions of f for which $I_0(a,t) = I_0(a,t)$ for all t, then $I_0 = I_1$.

Proof. To prove (i), set $z = y = x$ in 1 (ii). To prove (ii), set $z = x$ in 1 (i) and apply part (i). To prove (iii), let x and y be real numbers. Then by 1 (ii) and by part (ii), $I_0(x,y) = I_0(x,a) + I_0(a,y) = -I_0(a,x) + I_0(a,y)$. Similarly, $I_1(x,y) = I_1(x, a) + I_1(a,y) = -I_1(a,x) + I_1(a,y)$. Applying the hypothesis, we see at once that $I_0(x,y) = I_1(x,y)$. This completes the proof. //

Theorem 3. Let I be an integral function of f. Then for all x,y in R, we have $/I(x,y)/ \leq \beta/y-x/$.

Proof. Since $/f(t)/ \leq \beta$, for all real numbers t, we have $-B \leq f(t) - B$, for all t. Hence, by definition 1 (i), (3.1) $-B(y-x) \leq I(x,y) \leq B(y-x)$, if $x - y$. Hence

$$
\begin{aligned}
/I(x,y)/ &\leq B(y-x) \\
&= B/y-x/.
\end{aligned}
$$

Therefore the conclusion holds, if $x < y$. Now exchanging the roles of x and y in the preceding argument, we see that (3.2) $/I(y,x)/ = B/x - y/$, if $y < x$. Since $I(y,x) = (x, y)$, it follows immediately from (3.2) that the conclusion holds, if $y < x$.

Corollary 3.1 Let a be a fixed number and I an integral function of f. If $I'(a,x)$ exists, then $/I'(a,x)/ \leqslant B$.

Proof. By Definition 1 (ii), it follows that $I(a,x+h) = I(a,x) + I(x,x+h)$. Thus, by theorem 3, we have $/I(a,x+h) - I(a,x)/ \leqslant B/h/$. The conclusion follows easily from this. //

Theorem 4. Let I be an integral function of f. Then I satisfies a Lipschitz condition, i.e., there is a constant C such that, for all points p and q in RR^2, we have $/I(p) - I(q)/ \leqslant Cd(p,q)$, where $d(p,Q)$ is the distance between p and q.

Proof. Let $p = (x,y)$ and $q = (x^*, y^*)$. Then it follows from theorem 3 and definition 1 (ii) that

$$
\begin{aligned}
/I(p) - I(q)/ &= /I(x,y) - I(x^*, y^*)/ \\
&= /\{I(x,x^*) + I(x^*,y)\} - \{I(x^*, y) + I(y, y^*)\}/ \\
&\leqslant /I(x,x^*)/ + /I(y,y^*)/ \\
&\leqslant B/x - x^*/ + B/y - y^*/ \\
&\leqslant B\,D(p,q) + B\,d(p,q) \\
&= 2B\,d(p,q). \quad //
\end{aligned}
$$

Corollary 4. 1. The function I is uniformly continuous.

Proof. Let $\epsilon > 0$. We shall prove that there is a $\delta > 0$ such that for all points p and q in RR^2 $/I(p) - I(q)/ < \epsilon$, whenever $d(p,q) < \delta$. It is easy to verify that $\delta = \epsilon / C$, where C is the constant in theorem 4, will suffice. //

Theorem 5 (Lang [2, p. 213]). Let I be an integral function of f and let a be a fixed real number. If f is continuous at x, then $f(x) = I'(a,x)$.

Proof. Note that the function $I(a,x)$ is a function of the variable x alone. For each $\delta > 0$, define

$$
\begin{aligned}
m(x,\delta) &= \inf \{ f(t) \; . \; /x - t/ \leqslant \delta \} \\
M(x,\delta) &= \sup \{ f(t) : /x - t/ \leqslant \delta \}
\end{aligned}
$$

Claim $f(x) = \lim M(x,\delta) = \lim m(x,\delta)$, as $\delta \to 0$.

Clearly $f(x) \leq M(x, \delta)$, for each δ . Let $\in > 0$. Since f is continuous at x, there is a $\delta > 0$ such that $/f(x) - f(t)/ \quad \delta$. whenever $/x - t/ < \delta$ *. Thus, if $0 < \delta < \delta$., we have $f(x) \leq M(X,\delta) \leq f(x) + \in$. Hence, as $\delta \to 0$, we obtain

$$f(x) \leq \lim M(x, \delta) \leq f(x) + \in .$$

Since $\in$ is arbitrary, it follows that $\lim M(x,\delta) = f(x)$, as $\delta \to 0$. By analogy, we also have $\lim m(x,\delta) = f(x)$. This proves the claim.

Now, if $0 < h \leq \delta$, then we have $m(x, \delta) \leq f(t) \leq M(x, \delta)$ for $x \leq t \leq x + h$. It follows from definition 1 (i) that (5.1) $m(x, \delta) \leq \frac{1}{-h} I(x,x+h) \leq M(x,\delta)$

On the other hand, if $-\delta \leq h < 0$, then we also have (5.2) $m)x, \delta) \leq \frac{1}{h} I(x + h, x) \leq M(x;\delta)$. Combining (5.1) and (5.2), we see that (5.1) holds if $0 \leq /h/ \leq \delta$. Letting $\delta \to 0$, we obtain $f(x). \leq \lim_{h \to 0} \frac{1}{h} I(x,x+h) \leq f(x)$

The conclusion follows easily from the last inequality because $I(a,x + h) = I(a,x) = I(x,x+h)$ //

Theorem 6. Suppose $F'(t) = f(t)$, for all t. Define $I(x,y) - F(y) - F(x)$ for all x, y in RR^2. Then I is an integral function of f.

Proof. Suppose $x < y$ and suppose $m \leq f(t) \leq M$, for all t in $[x,y]$. Then, according to the mean-value theorem, there is a point c between x and y such that

$$\frac{F(y) - F(x)}{y - x} = F'(c).$$

Since $F'(c) = f(c)$, we have $m \leq F'(c) \leq M$. Therefore,

$$m(y - x) \leq F(y) - F(x) \leq M(y - x).$$

The first part of definition 1 is thus verified. The second part is straightforward. For if x, y, and z are real numbers, then

$$I(x,z) = F(z) - F(x)$$

$$= F(z) - F(y) + F(y) - F(x)$$

$$= I(y,z) + I(x,y)$$

$$= I(x,y) + I(y,z). \quad //$$

Theorem 5 and theorem 6 are our version of the fundamental theorem of calculus. Note that theorem 5 fails if we omit the continuity assumption on f. For example, suppose $f(x)=1$, if x is positive, and 0, if x is not positive. A simple calculation shows that $I(-1,x)=x$, if x is positive, and 0, if x is not positive. Clearly $I'(-1,x)$ does not exist, if $x=0$.

Theorem 7 (Lang [2 p. 224]). Let I_0 and I_1 be real-valued functions of two real variables given by

$$(7.1)\quad \begin{cases} I_0(x,y) = \underline{\int}_x^y f(t)\,dt \\ I_1(x,y) = \overline{\int}_x^y f(t)\,dt \end{cases}$$

Then I_0 and I_1 are integral functions of f. Moreover, if I is an integral function of f, then for all x, y with $x \leq y$, we have

$$(7.2)\quad I_0(x,y) \leq I(x,y) \leq I_1(x,y).$$

Proof. We prove that I_1 is an integral function of f. The proof for I_0 is similar.

Let u, v be real numbers with $u<v$, and suppose $m\leq f(t)\leq M$, for $u\leq t\leq v$. Let P $u=x_0<x_1<x_2<\cdots<x_n=v$, be a partition of $[u,v]$. Since $m\leq M_i\leq M$, for each i, we have

$$m(x_i - x_{i-1}) \leq M_i\,(x_i - x_{i-1}) \leq M\,(x_i - x_{i-1}).$$

Summing up, we obtain

$$m\,(v-u) \leq U(P) \leq M\,(v-u).$$

Since P is arbitrary, it follows that

$$m\,(v-u) \leq \overline{\int}_u^v f(t)\,dx \leq M(v-u).$$

Thus, property (i) of definition 1 is verified.

To verify the second property, let u, v, and w be real numbers with $u<v<w$. Let P be a partition of $[u,w]$. Then $Q=P\cup\{v\}$ is also a partition of $[u,w]$, which is finer than P. It follows that $U(Q)\leq U(P)$. But $Q=Q_1\cup Q_2$, where $Q_1 = Q \cap [u,v]$ and $Q_2=Q\cap[u,v]$. Therefore,

$$\begin{aligned} I_1(u,v) + I_1(v,w) &= \overline{\int}_u^v f(t)\,dt + \overline{\int}_v^w f(t)\,dt \\ &\leq U(Q_1)+U(Q_2) \\ &= U(Q) \\ &\leq U(P). \end{aligned}$$

Since P is arbitrary: it follows that

$$I_1(u,v) + I_1(v,w) \leqslant \bar{\int}_u^w f(t)\,dt = I_1(u,w).$$

To prove the reverse inequality, let $\alpha = I_1(u,v)$, and $\beta = I_1(v,w)$. Let $\epsilon > 0$. Then there is a partition P of $[u,v]$ such that $U(P) < \alpha + \epsilon$. Thus, $P \cup Q$ is a partition of $[u,w]$, and we have

$$\begin{aligned} I_1(u,w) &= \bar{\int}_u^w f(t)\,dt \\ &\leqslant U(P \cup Q) \\ &= U(P) + U(Q) \\ &< \alpha + \beta + 2\epsilon. \end{aligned}$$

Since ϵ is arbitrary, we have $I_1(u,w) \leqslant \alpha + \beta$.

The other cases are proved using the preceding case and the fact that $I_1(u,v) = I_1(v,u)$, for all real numbers u and v. For example, suppose $u \leqslant w \leqslant v$. Then, we have $I_1(u,v) = I_1(u,w) + I_1(w,v)$. Hence, $I_1(u,w) = I_1(u,v) - I_1(w,v) = I_1(u,v) + I_1(v,w)$.

To prove (7.2), let $P: x = x_0 < x_1 < \cdots < x_n = y$ be a partition of $[x,y]$. For each i, we have

$$m(x_i - x_{i-1}) \leqslant I(x_{i-1}, x_i) \leqslant M_i(x_i - x_{i-1}).$$

Summing up, we obtain

$$L(P) \leqslant I(x,y) \leqslant U(P).$$

Since P is arbitrary, (7.2) holds. //

Let g be a bounded real-valued function of a real variable and let a and b be real numbers with $a \leqslant b$. Define the function $g^* . \mathbb{R} \to \mathbb{R}$ by

$$g^*(t) = \begin{cases} g(t), & \text{if } a \leqslant t \leqslant b \\ 0, & \text{otherwise} \end{cases}$$

We say that g^* is <u>integrable,</u> if the integral function of g^* is unique. We say that g is integrable on $[a,b]$, if g^* is integrable.

Theorem 8. Let a and b be real numbers with $a \leqslant b$. A bounded function g is integrable on $[a,b]$, if and only if g is Riemann integrable on $[a,b]$, i.e., the Riemann integral $\int_a^b g(t)\,dt$ exists. In this case, if I is the integral function of g^*, we have

$$I(a,b) = \int_a^b g(t)\,dt.$$

Proof. Suppose g is integrable on [a,b]. Then the integral function of g* is unique, and it follows from theorem 7 that

$$\underline{\int}_x^y g^*(t)\,dt = \overline{\int}_x^y g^*(t)\,dt,$$

for all real numbers x and y. In particular, we have

$$\underline{\int}_a^b g^*(t)\,dt = \overline{\int}_a^b g^*(t)\,dt.$$

Since $g^*(t) = g(t)$, if $a \leqslant t \leqslant b$, it follows that

$$\underline{\int}_a^b g(t)\,dt = \overline{\int}_a^b g(t)\,dt.$$

Thus, by definition, the Riemann integral $\int_a^b g(t)\,dt$ exists, and is equal to the common value of the upper and lower integrals above.

Conversely, suppose the Riemann integral of g on [a,b] exists. We define

$$J_0(x,y) = \underline{\int}_x^y g^*(t)\,dt$$

$$J_1(x,y) = \overline{\int}_x^y g^*(t)\,dt$$

By theorem 7, J_0 and J_1 are integral functions of g^*. We shall show that these two integral functions are identical. In view of proposition 2 (iii), it is enough to show that $J_0(a,x) = J_1(a,x)$, for all x in RR. Suppose, first, that $x < a$. Since $g^*(t) = 0$, if $x \leqslant t < a$, it follows that $J_0(x,a) = 0 = J_1(x,a)$. Consequently, $J_0(a,x) = J_1(a,x)$ Next suppose that $a \leqslant x \leqslant b$. Since $g^*(t) = g(t)$, if $a \leqslant t \leqslant b$, and since g is Riemann integrable on [a,b], it follows that $J_0(a,x) = J_1(a,x)$, if $a \leqslant x \leqslant b$. Finally, suppose that $b < x$. Since $g^*(t) = 0$, if $b < t \leqslant x$, it follows that $J_0(b,x) = 0 = J_1(b,x)$. Thus $J_0(a,x) = J_0(a,b) + J_0(b,x) = J_0(a,b)$. Similarly, $J_1(a,x) = J_1(a,b)$. But $J_0(a,b) = J_1(a,b)$, by the preceding case. Hence $J_0(a,x) = J_1(a,x)$. This completes the proof of the theorem. //

Theorem 9. If f is continuous on [a,b], then f is integrable on [a,b].

Proof. We can prove this theorem using Theorem 8 and a well-known result from calculus that says. If f is continuous on [a,b], then f is Riemann integrable on [a,b]. However, it is preferable to give a proof that doesn't depend on results about Riemann integrals.

Let $f^*(t) = f(t)$, if $a \leqslant t \leqslant b$, and $f^*(t) = 0$, otherwise. It follows from the hypothesis that f^* is continuous at each point in the set $S = RR\ \{a, b\}$. Now let I_0 and I_1 be integral functions of f^*. Then by theorem 5, $I'_0(a,t) = I'_1(a,t) = f^*(t)$, for all t in S. Using the mean-value theorem, we obtain $I_0(a,t) = I_1(a,t) + C$, for all t in S, for some constant C. This last equation actually holds for all t, since I_0 and I_1 are continuous and since S is dense in RR. Letting $t = a$, we see that $C = 0$, and consequently, it follows that $I_0(a,t) = I_1(a,t)$, for all t in RR. Therefore, by proposition 2 (iii), $I_0 = I_1$. We have shown that the integral function of f* is unique, i.e., f is integrable on [a,b]. //

Application

Suppose a particle, moving along the x-axis, is acted on by a force f(x) depending on the position x of the particle. We denote the work done between a and b by W(a,b). It is reasonable that the work done should satisfy the following properties:

(i) If a, b, c are three numbers, with $a \leq b \leq c$, then $W(a,c) = W(a,b) + W(b,c)$.

(ii) If g is a stronger force than f on the interval [a,b], then we shall do more work with g than with f. In particular, if $m \leq f(x) \leq M$, on the interval [a,b], then $m(b - a) \leq W(a,b) \leq M(b - a)$.

Therefore according to definition 1, W is an integral function of f. If f is integrable on [a,b], then it is also Riemann integrable on [a,b]. In this case we have

$$W(a,b) = \int_c^b f(x)\,dx.$$

Hence, we have obtained the usual formula for work in a straightforward manner. This example, which is lifted from Lang [2, p. 287], shows us one obvious merit of the function approach.

References

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