# LIMITS AND CONTINUITY OF FUNCTIONS

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The use of limits, though in uncrystallized form, can be traced to the ancient Greeks. For example, Archimedes had used the concept to find an approximation of the value of $2\pi$. Specifically, he obtained an approximate value by taking the "limit" of the perimeters of regular polygons inscribed in a circle of radius 1 as the number of sides of the polygons constructed increase without bound.

In this lecture, we shall study limits and continuity of functions. These concepts are essential to the main subjects that form the nucleus of calculus: the derivative and the integral.

**Intuitive idea of limit.** Let $f$ be a function given by $y = f(x)$ and defined at each $x$ on some interval $I$ containing $a$, except possibly at $a$ itself. When we say that "*L is the limit of f(x), as x approaches a*" we roughly say that $f(x)$ gets close to $L$ as x gets close to $a$. To see this, let us consider a couple of examples.

1.1.1 EXAMPLE. Let $f$ be defined by $y = f(x) = 5x + 2$. We shall in-

| $x$ | 1 | 0.5 | 0.25 | 0.10 | 0.01 | 0.001 | 0.0001 | 0.00001 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 7 | 4.5 | 3.25 | 2.5 | 2.05 | 2.005 | 2.0005 | 2.00005 |

**Table 1** ($x > 0$)

| $x$ | −1 | −0.5 | −0.25 | −0.10 | −0.01 | −0.001 | −0.0001 | −0.00001 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | −3 | −0.5 | 0.75 | 1.5 | 1.95 | 1.995 | 1.9995 | 1.99995 |

**Table 2** ($x < 0$)

vestigate the function values $f(x)$, when $x$ is close to zero but not equal to 0.

From Tables 1 and 2, we see that the value of $5x + 2$ gets close to 2 as $x$ gets close to 0. Hence, we say that 2 is the limit of $5x + 2$ as $x$ approaches to 0 and write

$$\lim_{x \to 0} 5x + 2 = 2. \quad \#$$

Note that in the above example the limit of the function $f(x) = 5x + 2$ as $x$ approaches 0 is equal to the value of the function at 0, i.e., equal to $f(0) = 2$. However, this is not always the case as seen in the following example.

1.1.2 EXAMPLE. Consider the function $f$ defined by

$$f(x) = \frac{x^2 - 4}{x - 2}.$$

Clearly, the value of the function at $x = 2$ is not defined ($f(2) = 0/0$). However, $f(x)$ has a limit as $x$ approaches to 2. To see this let us again construct two tables,

| $x$ | 3 | 2.5 | 2.1 | 2.01 | 2.001 | 2.001 | 2.0001 | 2.00001 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 5 | 4.5 | 4.1 | 4.01 | 4.001 | 4.001 | 4.0001 | 4.00001 |

**Table 3** ($x > 2$)

| $x$ | 1 | 1.5 | 1.9 | 1.99 | 1.999 | 1.9999 | 1.99999 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 3 | 3.5 | 3.9 | 3.99 | 3.999 | 3.9999 | 3.99999 |

**Table 4** ($x < 2$)

We see from both tables above that $f(x)$ gets close to 4 as $x$ gets close to 2. Thus, 4 is the limit of $f(x)$ as $x$ approaches 2. #

In the above procedure, we may also specify the values of $f(x)$ first. In Example 1.1.1, we can actually make the value of $f(x)$ as close to 2 as we please by taking $x$ close enough to 0. In other words, we can make the absolute difference (distance) between $f(x)$ and 2, written $|f(x) - 2|$, small by making the absolute difference between $x$ and 0 (or simply the absolute value of $x$ or $|x|$), small enough. This relationship is usually described by using two Greek letters: ε (epsilon) and δ (delta). Thus, we say that $|f(x) - 2|$ is less than a given positive number ε whenever $|x - 0| = |x|$ is less than some appropriately chosen positive number δ. It is worth noting that the value of δ is dependent on the value of ε.

**Formal definition.** Let us now give the standard definition of a limit.

1.2 DEFINITION. Let $f$ be a function which is defined at all $x$ on the open interval $I$ containing $a$, except possibly at $a$ itself. The ***limit of $f(x)$ as $x$ approaches to $a$*** is $L$, written

$$\lim_{x \to a} f(x) = L,$$

if for any $\varepsilon > 0$, however small, there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Note that in the above definition, $L$ is the limit of $f(x)$ as $x$ approaches $a$ either from the right or from the left.

The following example uses the definition to prove that a given function has the indicated limit.

1.2.1 EXAMPLE. Prove that $\lim_{x \to 5} (2x - 7) = 3$.

*Solution.* We need to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|(2x - 7) - 3| < \varepsilon \text{ whenever } 0 < |x - 5| < \delta.$$

Now, $|(2x-7)-3| = |2x-10| = 2|x-5|$. Thus, we must show that

$$2|x-5| < \varepsilon \text{ whenever } 0 < |x-5| < \delta,$$

or, equivalently,

$$|x-5| < \frac{1}{2}\varepsilon \quad \text{whenever} \quad 0 < |x-5| < \delta.$$

If we set $\delta = \frac{1}{2}\varepsilon$, then we have

$$2|x-5| < 2\delta \quad \text{whenever} \quad 0 < |x-5| < \delta$$

or, equivalently,

$$|(2x-7)-3| < \varepsilon \quad \text{whenever} \quad 0 < |x-5| < \delta.$$

This will prove that

$$\lim_{x \to 5}(2x-7) = 3. \quad \#$$

The next theorem says that a function cannot approach two different limits at the same time. More precisely, if the limit of a function exists, then this limit must be unique.

1.3 THEOREM. (UNIQUENESS OF A LIMIT) *If we have* $\lim_{x \to a} f(x) = L_1$ *and* $\lim_{x \to a} f(x) = L_2$, *then* $L_1 = L_2$.

*Proof.* Suppose $L_1 \neq L_2$ and let $\varepsilon = |L_1 - L_2|$. Since $L_1$ is a limit of $f(x)$, as $x$ approaches $a$, there exists a $\delta_1 > 0$ such that

$$|f(x) - L_1| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < |x-a| < \delta_1.$$

Also, since $L_2$ is a limit of $f(x)$ as $x$ approaches $a$, there is $\delta_2 > 0$ such that

$$|f(x) - L_2| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < |x-a| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, applying the triangle inequality, we have

$$|L_1 - L_2| \le |L_1 - f(x)| + |f(x) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ when } 0 < |x - a| < \delta.$$

Hence,

$$|L_1 - L_2| < |L_1 - L_2|.$$

This is a contradiction; hence, our assumption is false. Therefore, $L_1 = L_2$, and the theorem is proved. #

**Limit theorems.** Finding limits by direct application of the definition is quite tedious. Hence, in order to evaluate limits of functions in a straight-forward manner, we shall need some powerful rules.

1.4 THEOREM. *If $b$ and $c$ are constants, then* $\lim_{x \to a} (bx + c) = ba + c$.

*Proof.* Let $\varepsilon > 0$. We want to show that there exits $\delta > 0$ such that

$$|(bx + c) - (ba + c)| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

To do this, we consider two cases:

Case 1. Suppose $b \neq 0$. Since $|(bx + c) - (ba + c)| = |b||x - a|$, we want to find a number $\delta > 0$ such that

$$|b||x - a| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta,$$

or, equivalently,

$$|x - a| < \frac{1}{|b|}\varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

If we choose $\delta = \varepsilon / |b|$, then we have

$$|(bx + c) - (bx + c)| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

This proves the theorem for Case 1.

Case 2. Suppose $b = 0$. If $b = 0$, then $|(bx + c) - (ba + c)| = |c - c| = 0$ for all $x$. Since $\varepsilon > 0$, we can take any positive number $\delta$. This proves the theorem for Case 2. #

1.4.1 EXAMPLE. $\lim\limits_{x \to -2} (7x - 8) = 7(-2) - 8 = -22$. #

1.4.2 COROLLARY (LIMIT OF A CONSTANT FUNCTION). *If $c$ is a constant, then for any real number $a$,*

$$\lim_{x \to a} c = c.$$

*Proof.* This follows from Theorem 1.4 by setting $b = 0$. #

1.4.3 EXAMPLE. $\lim\limits_{x \to 10} \sqrt{5} = \sqrt{5}$. #

1.4.4 COROLLARY (LIMIT OF THE IDENTITY FUNCTION). *For any real number $a$,*

$$\lim_{x \to a} x = a.$$

*Proof.* This follows from Theorem 1.4 by setting $b = 1$ and $c = 0$. #

1.4.5 EXAMPLE $\lim\limits_{x \to -\frac{2}{3}} x = -\frac{2}{3}$. #

1.5 THEOREM (LIMIT OF A SUM). *If* $\lim\limits_{x \to a} f(x) = L$, *and* $\lim\limits_{x \to a} g(x) = M$, *then* $\lim\limits_{x \to a} (f(x) + g(x)) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = L + M$.

*Proof.* Let $\varepsilon > 0$. Then, by definition of $L$, there is a $\delta_1 > 0$ such that

$$|f(x) - L| < \varepsilon/2 \quad \text{whenever } 0 < |x - a| < \delta_1.$$

Similarly, there exists a $\delta_2 > 0$ such that

$$|g(x) - M| < \varepsilon/2 \quad \text{whenever} \quad 0 < |x - a| < \delta_2.$$

Now, let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$. Hence,

$$|f(x) - L| < \varepsilon/2 \quad \text{whenever} \quad 0 < |x - a| < \delta \quad \text{and}$$

$$|g(x) - M| < \varepsilon/2 \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

It follows that

$$|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

whenever $0 < |x - a| < \delta$. This proves the theorem. #

The following theorem is an extension of Theorem 1.5.

1.6 THEOREM. *If* $\lim_{x \to a} f_1(x) = L_1$, $\lim_{x \to a} f_2(x) = L_2$, $\cdots$ $\lim_{x \to a} f_n(x) = L_n$, *then we have*

$$\lim_{x \to a} \left[ f_1(x) + f_2(x) + \cdots + f_n(x) \right] = L_1 + L_2 + \cdots + L_n.$$

*Proof.* This is proved by using Theorem 1.5 and mathematical induction. #

1.6.1 EXAMPLE. From Corollary 1.4.2, and Corollary 1.4.4, we have

$$\lim_{x \to 5} 7 = 7, \text{ and } \lim_{x \to 5} x = 5.$$

Therefore, from Theorem 1.5, $\lim_{x \to 5}(x + 7) = 5 + 7 = 12$. #

The proofs of the following important theorems are found in standard calculus texts, e.g., see [2].

1.7 THEOREM (LIMIT OF A PRODUCT). *If* $\lim_{x \to a} f_1(x) = L_1$, *and* $\lim_{x \to a} f_2(x) = L_2$, *then* $\lim_{x \to a} \left[ f_1(x) f_2(x) \right] = L_1 L_2$.

Theorem 1.7 can be extended to any finite number of functions by applying mathematical induction.

1.8 THEOREM. *If* $\lim_{x \to a} f_1(x) = L_1$, $\lim_{x \to a} f_2(x) = L_2$, $\cdots$ $\lim_{x \to a} f_n(x) = L_n$, *then we have*

$$\lim_{x \to a} \left[ f_1(x) f_2(x) \cdots f_n(x) \right] = L_1 L_2 \cdots L_n.$$

1.8.1 EXAMPLE. From Theorem 1.4, we have: $\lim_{x \to -2} (7x + 11) = -3$ and $\lim_{x \to -2} (-4x + 1) = 9$. Thus, from Theorem 1.7, it follows that

$$\lim_{x \to -2} (7x + 11)(-4x + 1) = \lim_{x \to -2} (7x + 11) \cdot \lim_{x \to -2} (-4x + 1) = -27. \quad \#$$

1.9 THEOREM. *If* $\lim_{x \to a} f(x) = L$ *and* $n$ *is any positive integer, then we have*

$$\lim_{x \to a} [f(x)]^n = L^n.$$

1.9.1 EXAMPLE. By Theorem 1.4, $\lim_{x \to \frac{1}{2}} (7x - 3) = \frac{1}{2}$. Thus, it follows from Theorem 1.9 that

$$\lim_{x \to \frac{1}{2}} (7x - 3)^5 = \left( \lim_{x \to \frac{1}{2}} (7x - 3) \right)^5 = \left( \tfrac{1}{2} \right)^5 = \frac{1}{32}. \quad \#$$

1.10 THEOREM. *If* $\lim_{x \to a} f(x) = L$, *then* $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$ *provided* $L > 0$ *and* $n$ *is a positive integer, or* $L < 0$ *and* $n$ *is a positive odd integer.*

1.10.1 EXAMPLE. By Theorem 1.4, $\lim_{x \to 4} (2x - 9) = -1$; by Theorem 1.9, $\lim_{x \to 4} (2x - 9)^3 = (-1)^3 = -1$. Hence, by Theorem 1.10, it follows that

$$\lim_{x \to 4} \sqrt[5]{(2x-9)^3} = \sqrt[5]{\lim_{x \to 4} (2x-9)^3} = \sqrt[5]{-1} = -1. \quad \#$$

1.11 THEOREM. *If* $\lim_{x \to a} f(x) = L$ *and* $\lim_{x \to a} g(x) = M$, $M \neq 0$, *then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

1.11.1 EXAMPLE. By Corollary 1.4.4 and Theorem 1.9, $\lim_{x \to 3} x^3 = 27$; also by Theorem 1.2, $\lim_{x \to 3} (2x + 3) = 9$. Thus, by Theorem 1.11, we have

$$\lim_{x \to 3} \frac{x^3}{2x+3} = \frac{\lim_{x \to 3} x^3}{\lim_{x \to 3} (2x+3)} = \frac{27}{9} = 3. \quad \#$$

In attempting to evaluate $\lim_{x \to a} \frac{f(x)}{g(x)}$, we frequently encounter situations in which $\lim_{x \to a} g(x) = 0$. In such cases, Theorem 1.11 does not apply. But the following theorem is often applicable.

1.12 THEOREM. *If* $F$ *and* $G$ *are two functions such that* $F(x) = G(x)$ *for all* $x \neq a$, *and if* $\lim_{x \to a} G(x)$ *exists, then*

$$\lim_{x \to a} F(x) = \lim_{x \to a} G(x).$$

*Proof.* By hypothesis, $\lim_{x \to a} G(x) = L$ exists. Therefore, for any $\varepsilon > 0$, there exits a $\delta > 0$ for which

$$|G(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta,$$

where $x$ is in the domain of the function $G$.

Again, by hypothesis, $G(x) - L = F(x) - L$ for all $x \neq a$. Thus we easily see that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then

$$|F(x) - L| = |G(x) - L| < \varepsilon.$$

Thus, $\lim_{x \to a} F(x) = L$, and the theorem is proved. #

1.12.1 EXAMPLE. Evaluate the following limits:

(a) $\lim_{x \to 2} \dfrac{x^3 - 8}{x - 2}$ (b) $\lim_{x \to 0} \dfrac{\sqrt{x+1} - 1}{x}$

*Solution:* (a) Since $\lim_{x \to 2} (x - 2) = 0$, Theorem 1.11 is not applicable. Now observe that if $x \neq 2$, then

$$F(x) = \frac{x^3 - 8}{x - 2} = \frac{(x-2)(x^2 + 2x + 4)}{(x - 2)} = x^2 + 2x + 4 = G(x).$$

Thus, by Theorem 1.12 and the other limit theorems, we have

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} (x^2 + 2x + 4) = (2)^2 + 2(2) + 4 = 12. \quad \#$$

(b) Again, Theorem 1.11 cannot be used to evaluate the given limit. So, let

$$H(x) = \frac{\sqrt{x+1} - 1}{x}.$$

If $x \neq 0$, then by rationalizing the numerator of $H(x)$, we have

$$H(x) = \frac{(\sqrt{x+1}-1)}{x} \cdot \frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)} = \frac{x}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1} = G(x).$$

Therefore, by Theorem 1.12, we have

$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}. \quad \#$$

**One-sided limits.** Observe that when considering $\lim_{x \to a} f(x)$ we are concerned with values of $x \neq a$ in an open interval $I$ containing $a$, which are close to $a$. However, the function $f$ may not be defined in any open interval containing $a$. In such a case, we are led to consider either those values of $x$ greater than $a$ or values of $x$ less than $a$. If the function $f$ is defined everywhere in some interval $(a, c)$ and if the value of $f(x)$ can be made as close to the number $L$ as we please by taking $x$ in this interval close enough to $a$, then we say that the "*limit of f(x) as x approaches a from the right is L*" and write

$$\lim_{x \to a^+} f(x) = L.$$

On the other hand, if $f$ is defined everywhere in an open interval $(b,a)$ and $f(x)$ can be made as close to the number $L$ as we please by taking values of $x$ sufficiently close to $a$, then the "*limit of f(x) as x approaches a from the left is L*" and we write

$$\lim_{x \to a^-} f(x) = L.$$

As an illustration, consider the function $f$ defined by $f(x) = \sqrt{x-1}$. Note that if $x < 1$, then $f(x)$ is not a real number (hence, $f(x)$ "does not exist"). Thus, we cannot consider the ordinary limit $\lim_{x \to 1} \sqrt{x-1}$. However, if we consider values of $x$ that are greater than 1, then we see that the value of $\sqrt{x-1}$ can be made close enough to 0 by taking $x$ sufficiently close to 1. In this case, we write

$$\lim_{x\to 1^+} \sqrt{x-1} = 0.$$

We shall now formally define one-sided limits.

1.13.1 DEFINITION. (RIGHT-HAND LIMIT) Let $f$ be a function which is defined at every number in some open interval $(a,c)$. Then the ***limit of $f(x)$ as $x$ approaches $a$ from the right*** is $L$, written

$$\lim_{x\to a^+} f(x) = L,$$

if for every $\varepsilon > 0$, however small, there exists a $\delta > 0$ such that

$$|f(x)-L| < \varepsilon \quad \text{whenever} \quad 0 < x-a < \delta.$$

1.13.2 DEFINITION (LEFT-HAND LIMIT) Let $f$ be a function which is defined at every number in some open interval $(d,a)$. Then the ***limit of $f(x)$ as $x$ approaches $a$ from the left*** is $L$, written

$$\lim_{x\to a^-} f(x) = L,$$

if for every $\varepsilon > 0$, however small, there is a $\delta > 0$ such that

$$|f(x)-L| < \varepsilon \quad \text{whenever} \quad 0 < a-x < \delta.$$

REMARK. The limit theorems discussed earlier still hold if "$x \to a$" is replaced by "$x \to a^+$" or "$x \to a^-$".

1.13.3 EXAMPLE. Evaluate $\displaystyle\lim_{x\to 1^+} \frac{x-1}{\sqrt{x^2-1}}$.

*Solution.* We apply an analogue of Theorem 1.12 to evaluate the given limit. Notice that if $x \neq 1$, then

$$F(x) = \frac{x-1}{\sqrt{x^2-1}} = \frac{(x-1)}{\sqrt{(x+1)(x-1)}} \cdot \frac{\sqrt{(x+1)(x-1)}}{\sqrt{(x+1)(x-1)}}$$

$$= \frac{(x-1)\sqrt{x^2-1}}{\sqrt{(x+1)^2(x-1)^2}} = \frac{(x-1)\sqrt{x^2-1}}{(x+1)(x-1)} = \frac{\sqrt{x^2-1}}{(x+1)} = G(x).$$

It follows from the previous limit theorems that

$$\lim_{x\to 1^+} \frac{x-1}{\sqrt{x^2-1}} = \lim_{x\to 1^+} \frac{\sqrt{x^2-1}}{(x+1)} = \frac{\sqrt{(1)^2-1}}{(1+1)} = \frac{0}{2} = 0. \quad \#$$

The following theorem gives the relationship between the ordinary two-sided limit and one-sided limits.

1.14 THEOREM. *$\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist and are equal. Moreover,*

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

1.14.1 EXAMPLE. Does $\lim_{x\to 1} \frac{x-1}{|x-1|}$ exist?

*Solution.* To answer this question, we evaluate the one-sided limits of the given function at $x = 1$. First, observe that $|x-1| = x-1$, if $x > 1$, and $|x-1| = -(x-1)$, if $x < 1$. Hence,

$$\lim_{x\to 1^+} \frac{x-1}{|x-1|} = \lim_{x\to 1^+} \frac{x-1}{x-1} = 1 \quad \text{and}$$

$$\lim_{x\to 1^-} \frac{x-1}{|x-1|} = \lim_{x\to 1^-} \frac{x-1}{-(x-1)} = -1.$$

Thus, by Theorem 1.14, the ordinary limit $\lim_{x\to 1} \frac{x-1}{|x-1|}$ does not exist. #

**Infinite limits.** Consider the function $f$ defined by $f(x) = \frac{2}{(x-1)^2}$.
We investigate the values of $f$ when $x$ is close to 1. To do this let us look at the following tables.

| $x$ | 3 | 2 | 1.1 | 1.01 | 1.001 |
|---|---|---|---|---|---|
| $f(x)$ | 0.5 | 2 | 200 | 20,000 | 2,000,000 |

**Table 5** ( $x > 1$ )

From Table 5, we see that as $x$ gets close to 1 through values greater than 1, $f(x)$ increases without bound. In other words, we can make $f(x)$ as large as we please by taking $x$ close enough to 1. In this case we write

$$\lim_{x \to 1^+} \frac{2}{(x-1)^2} = +\infty.$$

| $x$ | −1 | 0 | 0.9 | 0.99 | 0.999 |
|---|---|---|---|---|---|
| $f(x)$ | 0.5 | 2 | 200 | 20,000 | 2,000,000 |

**Table 6** ( $x < 1$ )

From the second table we also see that $f(x)$ increases without bound as $x$ approaches 1 through values less than 1. Hence, we write

$$\lim_{x \to 1^-} \frac{2}{(x-1)^2} = +\infty.$$

Therefore, as $x$ approaches 1 from either the right or the left, $f(x)$ increases without bound. In this case we write

$$\lim_{x \to 1} \frac{2}{(x-1)^2} = +\infty.$$

1.15.1 DEFINITION. Let $f$ be a function which is defined at every number in some open interval $I$ containing $a$, except possibly at the number $a$ itself. We say that $f(x)$ ***increases without bound as** $x$ **approaches** $a$*, written

$$\lim_{x \to a} f(x) = +\infty,$$

if for any positive number $N$, there exists a $\delta > 0$ such that $f(x) > N$ whenever $0 < |x - a| < \delta$.

REMARK. The above limit can be read as "*the limit of f(x) as x approaches a is positive infinity.*" In such a case, the limit does not exist.

Similarly, we have the following definition.

1.15.2 DEFINITION. Let $f$ be a function which is defined at every number in some open interval $I$ containing $a$, except possibly at the number $a$ itself. We say that "$f(x)$ ***decreases without bound as*** $x$ ***approaches*** $a$," written

$$\lim_{x \to a} f(x) = -\infty,$$

if for every negative number $N$, there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - a| < \delta$.

1.16 THEOREM. *If $r$ is a positive integer, then*

(a) $\displaystyle\lim_{x \to 0^+} \frac{1}{x^r} = +\infty$ (b) $\displaystyle\lim_{x \to 0^-} \frac{1}{x^r} = \begin{cases} -\infty, & \text{if } r \text{ is odd} \\ +\infty, & \text{if } r \text{ is even.} \end{cases}$

1.16.1. EXAMPLE. (a) $\displaystyle\lim_{x \to 0^+} \frac{1}{x^6} = +\infty$ (b) $\displaystyle\lim_{x \to 0^-} \frac{1}{x^{11}} = -\infty$

1.17 THEOREM *If $a \in \mathbf{R}$, and if $\lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} f(x) = c$, where $c$ is a constant not equal to zero, then*

(*i*) *if $c > 0$ and if $g(x) \to 0$ through positive values of $g(x)$,*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = +\infty;$$

(*ii*) if $c > 0$ and if $g(x) \to 0$ through negative values of $g(x)$,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = -\infty;$$

(*iii*) if $c < 0$ and if $g(x) \to 0$ through positive values of $g(x)$,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = -\infty;$$

(*iv*) if $c < 0$ and if $g(x) \to 0$ through negative values of $g(x)$,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = +\infty.$$

1.17.1 EXAMPLE. Evaluate the following limits.

(a) $\lim\limits_{x \to 2^+} \dfrac{3x+1}{x^2-2x}$ (b) $\lim\limits_{x \to 1^-} \dfrac{-5x}{1-x}$ (c) $\lim\limits_{x \to 0} \dfrac{-2\sqrt{5-x}}{x^3}$

*Solution.* (a) Let $f(x) = 3x + 1$ and $g(x) = x^2 - 2x$. Then

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3x + 1 = 7 \text{ and } \lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} x^2 - 2x = 0.$$

Further, since $g(x)$ tends to zero through positive values of $g(x)$, by Theorem 1.17 (*i*), we have

$$\lim_{x \to 2^+} \frac{3x+1}{x^2-2x} = +\infty.$$

(b) Let $f(x) = -5x$ and $g(x) = 1 - x$. Then

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -5x = -5 \text{ and } \lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} 1 - x = 0.$$

Since $g(x) \to 0$ through positive values of $g(x)$, by Theorem 1.17 (*iii*),

$$\lim_{x \to 1^-} \frac{-5x}{1-x} = -\infty.$$

(c) Let $f(x) = -2\sqrt{5-x}$ and $g(x) = x^3$. Then

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -2\sqrt{5-x} = -2\sqrt{5} \text{ and}$$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} x^3 = 0.$$

Since $g(x) \to 0$ through negative values of $g(x)$, by Theorem 1.17 (*iv*),

$$\lim_{x \to 0^-} \frac{-2\sqrt{5-x}}{x^3} = +\infty. \quad \#$$

1.18 THEOREM. (*i*) *If* $\lim_{x \to a} f(x) = +\infty$ *and* $\lim_{x \to a} g(x) = c$, *where* $c$ *is any constant, then*

$$\lim_{x \to a} [f(x) + g(x)] = +\infty.$$

(*ii*) *If* $\lim_{x \to a} f(x) = -\infty$, *and* $\lim_{x \to a} g(x) = c$, *where* $c \in \mathbf{R}$, *then*

$$\lim_{x \to a} [f(x) + g(x)] = -\infty.$$

1.18.1 EXAMPLE. Evaluate the following limits:

(a) $\lim_{x \to 3^-} \left[\frac{-3}{3-x} + \frac{x}{2+x}\right]$ (b) $\lim_{x \to -1^+} \left[\frac{-5x}{-1-x} - \frac{4}{3x}\right]$

*Solution.* (a) Because $\lim_{x \to 3^-} \frac{-3}{3-x} = -\infty$ and $\lim_{x \to 3^-} \frac{x}{2+x} = \frac{3}{5}$, it follows from Theorem 1.18 (*ii*) that

$$\lim_{x\to 3^-}\left[\frac{-3}{3-x}+\frac{x}{2+x}\right]=-\infty.$$

(b) Because $\lim_{x\to -1^-}\frac{5x}{-1-x}=+\infty$ and $\lim_{x\to -1^-}\frac{4}{3x}=-\frac{4}{3}$, it follows from Theorem 1.18 (*i*) that

$$\lim_{x\to -1^-}\left[\frac{5x}{-1-x}-\frac{4}{3x}\right]=+\infty. \quad \#$$

1.19 THEOREM. *If* $\lim_{x\to a} f(x)=+\infty$ *and* $\lim_{x\to a} g(x)=c$, *where* $c$ *is any nonzero constant, then*

(*i*) *if* $c>0$, $\lim_{x\to a} f(x)\cdot g(x)=+\infty$;

(*ii*) *if* $c<0$, $\lim_{x\to a} f(x)\cdot g(x)=-\infty$.

1.19.1 EXAMPLE. Since $\lim_{x\to 2^-}\frac{-x}{x-2}=+\infty$ *and* $\lim_{x\to 2^-}\frac{1-x}{x+3}=-\frac{1}{5}$, it follows from Theorem 1.19 (*ii*) that

$$\lim_{x\to 2^-}\left[\frac{-x}{x-2}\cdot\frac{1-x}{x+3}\right]=-\infty. \quad \#$$

1.20 THEOREM. *If* $\lim_{x\to a} f(x)=-\infty$ *and* $\lim_{x\to a} g(x)=c$, *where* $c$ *is any nonzero constant, then*

(*i*) *if* $c>0$, $\lim_{x\to a} f(x)\cdot g(x)=-\infty$.

(*ii*) *if* $c<0$, $\lim_{x\to a} f(x)\cdot g(x)=+\infty$.

1.20.1 EXAMPLE. Because $\lim_{x\to 0^-}\frac{7}{x^3}=-\infty$ and $\lim_{x\to 0^-}\frac{4x-10}{x+5}=-2$, it follows from Theorem 1.20 (*ii*) that

$$\lim_{x\to 0}\left[\frac{7}{x^3}\cdot\frac{4x-10}{x-5}\right]=-\infty. \quad \#$$

**Continuity of a function.** In the previous section, we pointed out that if $\lim_{x\to a} f(x)$ exists, then its value is *not* necessarily equal to $f(a)$, the value of the function at the number $a$. In fact, $f(a)$ may be undefined. On the other hand, it is possible for $f(a)$ to exist and not $\lim_{x\to a} f(x)$. When both exist and are equal, we say that $f$ is continuous at $a$.

1.21 DEFINITION. The function $f$ is said to be ***continuous at the number*** $a$ if the following three conditions are satisfied:

(*i*) $f(a)$ exists;

(*ii*) $\lim_{x\to a} f(x)$ exists;

(*iii*) $\lim_{x\to a} f(x)=f(a)$.

If one or two of these conditions fail to hold, the function is said to be ***discontinuous at*** $x=a$.

Geometrically, a function is discontinuous at the number $a$ if there is a break in the graph of the function $f$ at the point where $x=a$.

Further, note that if $f$ is discontinuous at a number $a$, but for which $\lim_{x\to a} f(x)$ exists, then either $f(a)\neq\lim_{x\to a} f(x)$ or $f(a)$ does not exist. This discontinuity is called a ***removable discontinuity*** because we can redefine $f$ so that $f(a)=\lim_{x\to a} f(x)$ and, hence, $f$ becomes continuous at $a$. If the discontinuity is not removable, then it is called an ***essential discontinuity***.

1.21.1 EXAMPLE. Determine if the given function is continuous at the indicated number $a$. If it is discontinuous at $a$, determine if the discontinuity is removable or essential.

(a) $f(x)=\begin{cases}\dfrac{x-1}{|x-1|}, & \text{if } x\neq 1\\ 0, & \text{if } x=1\end{cases}$ ; $a=1$

*Solution.* Example 1.14.1 shows that $\lim_{x\to 1}\dfrac{x-1}{|x-1|}$ does not exist. Hence condition *(ii)* of Definition 1.21 is not satisfied. Therefore, $f$ is discontinuous at $a=1$. Moreover, the discontinuity is essential.

(b) $g(x)=\begin{cases}\dfrac{x^2-x-12}{x-4}, & \text{if } x\neq 4\\ 7, & \text{if } x=4\end{cases}$ ; $a=4$.

*Solution.* By definition of $g$, we see that $g(4)=7$. Now, in view of Theorem 1.12, we obtain

$$\lim_{x\to 4}\frac{x^2-x-12}{x-4}=\lim_{x\to 4}\frac{(x-4)(x+3)}{x-4}=\lim_{x\to 4}x+3=7.$$

Thus, $\lim_{x\to 4}g(x)=g(4)$. Thus, by Definition 1.21, $f$ is continuous at $x=4$.

(c) $h(x)=\dfrac{\sqrt[3]{x+1}-1}{x}$; $a=0$.

*Solution.* Using the special product $s^3-t^3=(s-t)(s^2+st+t^2)$, we see that for $x\neq 0$,

$$h(x)=\frac{\sqrt[3]{x+1}-1}{x}=\frac{(\sqrt[3]{x+1}-1)}{x}\cdot\frac{(\sqrt[3]{(x+1)^2}+\sqrt[3]{x+1}+1)}{(\sqrt[3]{(x+1)^2}+\sqrt[3]{x+1}+1)}$$

$$=\frac{x}{x(\sqrt[3]{(x+1)^2}+\sqrt[3]{x+1}+1)}$$

$$=\frac{1}{\sqrt[3]{(x+1)^2}+\sqrt[3]{x+1}+1}=F(x).$$

By Theorem 1.12,

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} F(x) = \lim_{x \to 0} \frac{1}{\sqrt[3]{(x+1)^2} + \sqrt[3]{x+1} + 1} = \frac{1}{3}$$

Since $h(0)$ is not defined, it follows that $h$ is discontinuous at 0. However, since $\lim_{x \to 0} h(x)$ exists we may redefine $h$ in such a way that $h(0) = 1/3$. Thus the discontinuity at 0 is removable. ♯

1.22 THEOREM. *If $f$ and $g$ are two functions which are continuous at the number $a$, then*

(*i*) *$f + g$ is continuous at $a$*

(*ii*) *$f - g$ is continuous at $a$*

(*iii*) *$f.g$ is continuous at $a$*

(*iv*) *$f/g$ is continuous at $a$, provided that $g(a) \neq 0$.*

**Limits at infinity**. In this section we shall investigate the function values $f(x)$ as $x$ increases (resp., decreases) without bound. Consider the function $f$ defined by

$$f(x) = \frac{x^2}{x^2+2}.$$

From the table below, we see that as $x$ increases through positive values, the function values $f(x)$ get closer and closer to 1. Intuitively, we see that we can make the value $f(x)$ as close to 1 as we please by taking $x$ large enough. In this case we write

| $x$ | 0 | 1 | 10 | 100 | 1000 |
|---|---|---|---|---|---|
| $f(x)$ | 0 | 0.33 | 0.980 | 0.9998 | 0.999998 |

**Table**

$$\lim_{x \to +\infty} \frac{x^2}{x^2+2} = 1.$$

The symbol "$x \to +\infty$" is used to denote that $x$ increases without bound. We now define the above concept formally.

1.23 DEFINITION. Let $f$ be a function which is defined at every number in some interval $(a, +\infty)$. The ***limit of $f(x)$ as $x$ increases without bound is $L$***, written

$$\lim_{x \to +\infty} f(x) = L,$$

if for any $\varepsilon > 0$, however small, there exists a number $N > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > N.$$

We now state the analogue of the preceding definition.

1.24 DEFINITION. Let $f$ be a function which is defined at every number in some interval $(-\infty, a)$. The ***limit of $f(x)$ as $x$ decreases without bound is $L$***, written

$$\lim_{x \to -\infty} f(x) = L,$$

if for any $\varepsilon > 0$, there exists a number $N < 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x < N.$$

The symbol "$x \to -\infty$" indicates that $x$ decreases without bound.

1.25 THEOREM. *If $r$ is a positive integer, then*

(a) $\displaystyle\lim_{x \to +\infty} \frac{1}{x^r} = 0$ (b) $\displaystyle\lim_{x \to -\infty} \frac{1}{x^r} = 0$

*Proof.* Let $f(x) = \dfrac{1}{x^r}$ and $L = 0$. We want show that for every $\varepsilon > 0$, there is there exists a number $N > 0$ such that

$$|f(x) - L| = \left|\frac{1}{x^r} - 0\right| < \varepsilon \quad \text{whenever} \quad x > N$$

or, equivalently,

$$\left|\frac{1}{x^r}\right| = \frac{1}{|x|^r} < \varepsilon \quad \text{whenever} \quad x > N$$

or, equivalently,

$$|x|^r > \frac{1}{\varepsilon} \quad \text{whenever} \quad x > N.$$

Since $r$ is a positive integer, this is equivalent to

$$|x| > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{r}} \quad \text{whenever} \quad x > N.$$

Thus, in order for the above to hold, we may choose $N = (\frac{1}{\varepsilon})^{\frac{1}{r}}$. That is,

$$\left|\frac{1}{x^r} - 0\right| < \varepsilon \quad \text{whenever} \quad x > N = (\frac{1}{\varepsilon})^{\frac{1}{r}}.$$

This proves $(i)$. We can prove $(ii)$ similarly. #

1.25.1 EXAMPLE. Evaluate the following limits:

(a) $\lim\limits_{x \to +\infty} \dfrac{3x^2 + 7}{2x^2 - 5}$ (b) $\lim\limits_{x \to -\infty} \dfrac{5x - 8}{\sqrt{10x^2 + 3}}$

*Solution.* (a) To apply Theorem 1.25, we divide both the numerator and the denominator by the highest power of $x$ occurring in either the numerator or the denominator, which in this case is $x^2$. Thus, applying Theorems 1.5, 1.11 and 1.25 (a), we have

$$\lim_{x\to+\infty}\frac{3x^2+7}{2x^2-5}=\lim_{x\to+\infty}\frac{3+\frac{7}{x^2}}{2-\frac{5}{x^2}}=\frac{\lim_{x\to+\infty}(3+\frac{7}{x^2})}{\lim_{x\to+\infty}(2-\frac{5}{x^2})}=\frac{3}{2}. \quad \#$$

(b) First, we divide the numerator and the denominator of the fraction by $\sqrt{x^2}$. Since we are considering negative values of $x$, then by definition, $\sqrt{x^2}=|x|=-x$. Thus, applying Theorems 1.5, 1.11, and 1.25 (b), it follows that

$$\lim_{x\to-\infty}\frac{5x-8}{\sqrt{10x^2+3}}=\lim_{x\to-\infty}\frac{-5+\frac{8}{x}}{\sqrt{10+\frac{3}{x}}}=\frac{-5}{\sqrt{10}}=-\frac{\sqrt{10}}{2}. \quad \#$$

**Limit theorems involving sine and cosine functions.** In the study of the derivatives of trigonometric functions, we shall need some important limit theorems. In particular, we shall need a theorem that is essential in the derivation of the formula for the derivative of the sine function. First, we state and prove the following important result.

1.26 THEOREM (SQUEEZE THEOREM). *Suppose that the functions $f$, $g$, and $h$ are defined on an open interval $I$ containing $a$ except possibly at $a$ itself and that $f(x)\le g(x)\le h(x)$ for all $x$ in $I$. Also, suppose that*

$$\lim_{x\to a} f(x)=L \text{ and } \lim_{x\to a} h(x)=L.$$

*Then $\lim_{x\to a} g(x)$ also exists and is equal to $L$.*

*Proof.* Let $\varepsilon>0$. Then there exist $\delta_1>0$ and $\delta_2>0$ such that

$|f(x)-L|<\varepsilon$ whenever $0<|x-a|<\delta_1$ and

$|h(x)-L|<\varepsilon$ whenever $0<|x-a|<\delta_2$.

Put $\delta=\min(\delta_1,\delta_2)$. Then

$-\varepsilon < f(x) - L < \varepsilon$ and $-\varepsilon < h(x) - L < \varepsilon$ whenever $0 < |x - a| < \delta$

Thus, by assumption, we have

$$-\varepsilon < f(x) - L \le g(x) - L \le h(x) - L < \varepsilon \text{ whenever } 0 < |x - a| < \delta,$$

i.e., $|g(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

This shows that $\lim_{x \to a} g(x) = L$. #

1.27 THEOREM. $\lim_{t \to 0} \frac{\sin t}{t} = 1$.

*Proof.* Let us first assume that $0 < t < \pi/2$. Figure 1 below shows the unit circle $x^2 + y^2 = 1$ and the sector BOP, where B is the point (1,0) and P is the point ($\cos t$, $\sin t$). The area $S$ of the circular sector BOP of radius $r = 1$ and central angle of radian measure $t$ is $S = (1/2)t$.

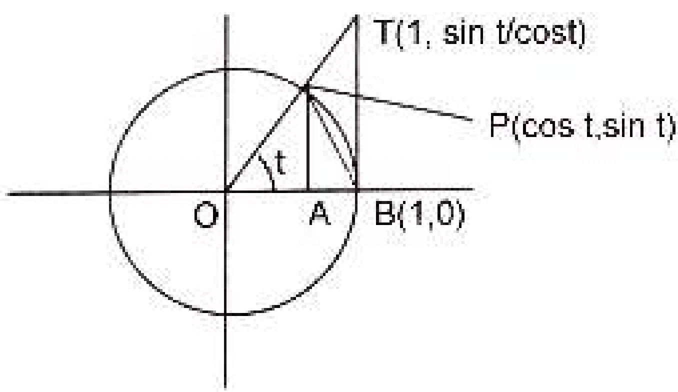


**Figure 1**

Next, consider the triangle BOP. The area of this triangle is

$$A_1 = (1/2)\, d(\text{A,P}).d(\text{O,B}) = (1/2) \sin t\,(1) = (1/2)\sin t.$$

The line through the points O(0,0) and P($\cos t$, $\sin t$) has slope

$$m = \frac{\sin t - 0}{\cos t - 0} = \frac{\sin t}{\cos t},$$

Thus, the equation of the line is $y = \frac{\sin t}{\cos t}x$.

If $A_2$ square units is the area of triangle BOT, then

$$A_2 = (\frac{1}{2}) \cdot d(\mathrm{B,T}) \cdot d(\mathrm{O,B}) = (\frac{1}{2}) \cdot \frac{\sin t}{\cos t} \cdot (1) = (\frac{1}{2}) \cdot \frac{\sin t}{\cos t}.$$

From Figure 1, we see that $A_1 < S < A_2$. That is,

$$\frac{1}{2}\sin t < \frac{1}{2}t < \frac{1}{2} \cdot \frac{\sin t}{\cos t}.$$

By assumption, $\sin t > 0$, and hence $2/\sin t > 0$. It follows that

$$1 < \frac{t}{\sin t} < \frac{1}{\cos t}.$$

Taking the reciprocal of each member of the above inequalities, we have

$$\cos t < \frac{\sin t}{t} < 1.$$

One of the above inequalities gives $\sin t < t$. Replacing $t$ by $(1/2)t$, we have $\sin (1/2)t < (1/2)t$. Using the identity

$$\frac{1 - \cos t}{2} = \sin^2 \frac{1}{2}t,$$

we have

$$\sin^2 \frac{1}{2}t = \frac{1 - \cos t}{2} < \frac{t^2}{4}.$$

The above inequality yields

$$1 - \frac{1}{2}t^2 < \cos t.$$

Since $0 < t < \pi/2$, we have

$$1 - \frac{1}{2}t^2 < \frac{\sin t}{t} < 1, \text{ if } 0 < t < \pi/2.$$

If $-\pi/2 < t < 0$, then $0 < -t < \pi/2$. Hence,

$$1 - \frac{1}{2}(-t)^2 < \frac{\sin(-t)}{-t} < 1, \text{ if } -\pi/2 < t < 0.$$

Thus,

$$1 - \frac{1}{2}t^2 < \frac{\sin t}{t} < 1, \text{ if } -\frac{1}{2}\pi < t < \frac{1}{2}\pi \text{ and } t \neq 0.$$

Since $\lim_{t\to 0}(1-\frac{1}{2}t^2) = 1$ and $\lim_{t\to 0} 1 = 1$, it follows from Theorem 1.26 that $\lim_{t\to 0}\frac{\sin t}{t} = 1$. This proves the theorem. #

1.27.1 EXAMPLE. Find $\lim_{x\to 0}\frac{\sin 7x}{\sin 2x}$, if it exists.

*Solution.* To apply Theorem 1.27, we write

$$\frac{\sin 7x}{\sin 2x} = \frac{7(\frac{\sin 7x}{7x})}{2(\frac{\sin 2x}{2x})} \quad \text{if } x \neq 0.$$

Since $\lim_{x\to 0}\frac{\sin 7x}{7x} = 1$ and $\lim_{x\to 0}\frac{\sin 2x}{2x} = 1$, we have

$$\lim_{x\to 0}\frac{\sin 7x}{\sin 2x} = \frac{7\lim_{x\to 0}\frac{\sin 7x}{7x}}{2\lim_{x\to 0}\frac{\sin 2x}{2x}} = \frac{7}{2}\cdot\frac{1}{1} = \frac{7}{2}. \quad \#$$

Theorem 1.27 can also be used to show that the sine and the cosine functions are continuous at zero.

1.28 THEOREM. *If $f(x) = \sin x$, then $f$ is continuous at* 0.

*Proof.* Let us show that the three conditions in Definition 1.21 are satisfied. Clearly, (*i*) $f(0) = \sin 0 = 0$. Now, writing $\sin t = t\frac{\sin t}{t}$, we have (*ii*) $\lim_{x \to 0} \sin x = \lim_{x \to 0} x \cdot \frac{\sin x}{x} = \lim_{x \to 0} x \cdot \lim_{x \to 0} \frac{\sin x}{x} = 0 \cdot 1 = 0.$

Hence, (*iii*) $\lim_{x \to 0} \sin x = \sin 0 = 0$. Therefore, $f$ is continuous at 0. #

1.29 THEOREM. *If* $f(x) = \cos x$, *then* $f$ *is continuous at* 0.

*Proof.* Again, we verify the three conditions given in Definition 1.21.

(*i*) $\cos 0 = 1$

(*ii*) $\lim_{x \to 0} \cos x = \lim_{x \to 0} \sqrt{1 - \sin^2 x} = \sqrt{\lim_{x \to 0}(1 - \sin^2 x)} = \sqrt{1 - 0} = 1.$

In (*ii*), we can replace $\cos x$ by $\sqrt{1 - \sin^2 x}$ because $\cos x > 0$ when $x \neq 0$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Thus,

(*iii*) $\lim_{x \to 0} \cos x = \cos 0 = 1.$

Therefore, $f$ is continuous at 0. #

The following theorem is also important. It is an immediate consequence of the previous three theorems.

1.30 THEOREM. $\lim_{t \to 0} \frac{1 - \cos t}{t} = 0.$

*Proof.*

$$\lim_{t \to 0} \frac{1 - \cos t}{t} = \lim_{t \to 0} \frac{(1 - \cos t)(1 + \cos t)}{t(1 + \cos t)}$$

$$= \lim_{t \to 0} \frac{(1 - \cos^2 t)}{t(1 + \cos t)}$$

$$= \lim_{t \to 0} \frac{\sin^2 t}{t(1 + \cos t)}$$

$$= \lim_{t \to 0} \frac{\sin t}{t} \cdot \lim_{t \to 0} \frac{\sin t}{(1 + \cos t)} = 1 \cdot \frac{0}{2} = 0.$$

This proves the theorem. #

1.30.1 EXAMPLE. Evaluate $\lim_{x \to 0} \frac{1 - \cos x}{\sin x}$, if the limit exists.

*Solution.* If $x \neq 0$, then

$$\frac{1 - \cos x}{\sin x} = \frac{\frac{1 - \cos x}{x}}{\frac{\sin x}{x}}.$$

Thus, applying Theorems 1.11, 1.27, and 1.30, we have

$$\lim_{x \to 0} \frac{1 - \cos x}{\sin x} = \frac{\lim_{x \to 0} \frac{1 - \cos x}{x}}{\lim_{x \to 0} \frac{\sin x}{x}} = \frac{0}{1} = 0. \quad \#$$

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