INTEGRATION AND SOME APPLICATIONS

Emmanuel M. Lagare

It has been the traditional view, albeit inaccurate, that differential calculus and integral calculus be treated separately. With such a perspective, teachers lose the pedagogical advantage of viewing these two facets of the calculus like two sides of the same coin and exploiting the synergy of both.

This intimate connection between derivatives and integrals was first discovered by Newton and Leibnitz and has formed the linchpin in the development of physics. It has also played a major role in the application of calculus to the real world.

The lecture will substantively cover only single integrals although multiple (particularly double) integrals will be briefly touched. The focus will be the concept of the sum and the role it plays in the application of integration.

Proofs will be generally omitted in these notes due to space limitations. However, you can find the proofs in the texts mentioned in the bibliography or any good calculus text which are now increasingly available in local reprints at the bookstores. The assumption throughout is that other concepts with bearing on integration have been discussed or will be discussed by other speakers.

Partitions and integration. The concept of partitions or divisions play a chief role in the development of the theory of integration. Depending on the specifications of partitions, we arrive at a different integral. What we shall be discussing will be partitions which will give rise to the Riemann integral.

3.1 DEFINITION. A ***partition*** D of $[a,b]$ is a collection of points $\{y_0, y_1, \dots, y_n\}$ such that $a = y_0 < y_1 < \cdots < y_n = b$. The ***norm*** $\|D\|$ of the partition D is the largest of the differences $(y_i - y_{i-1})$, $i = 1, 2, \dots, n$.

3.2 EXAMPLE. Let $a = 1$, $b = 4$. One such partition is

$$D: \ a = 1 < 1.2 < 1.5 < 1.8 < 2.1 < 2.5 < 3.0 < 3.4 < 3.6 < 3.8 < 4 = b$$

For this partition, the norm is $\|D\| = 3.0 - 2.5 = 0.5$. #

The next definition gives the requirements for a function f to be Riemann integrable. This definition was given by Riemann. An equivalent definition was given by Darboux.

3.3 DEFINITION. Let f be a function with domain the interval of real numbers $[a,b]$, and range which is a subset of the real numbers. We say that f is ***integrable*** on $[a,b]$ if we can find a number I with the property that given $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever a division D satisfies the conditions that $\|D\| < \delta$ and x_i is in $[y_{i-1}, y_i]$, for $i = 1, 2, \ldots, n$, we have

$$\left| \sum_{i=1}^{n} f(x_i)(y_i - y_{i-1}) - I \right| < \varepsilon.$$

We say that I is the (***Riemann***) ***integral*** or ***definite*** (***Riemann***) ***integral*** of f over $[a,b]$. This value is denoted by $\int_a^b f(x)dx$. In some sense, the value of I is given by

$$\int_a^b f(x)dx = \lim_{\|D\|\to 0} \sum_{i=1}^{n} f(x_i)(y_i - y_{i-1}).$$

When $f(x) \geq 0$, each term in the above summation is the area of the rectangle of width $y_i - y_{i-1}$ and length $f(x_i)$. As $\|D\| \to 0$ the widths become smaller and smaller and the sum of the rectangles better and better approximates the area under the curve of $f(x_i)$ from a to b.

3.4 EXAMPLE. Let $f(x) = 2x$ and D as given above. Choosing the x_i's as follows:

$x_1 = 1.2$, $x_2 = 1.3$, $x_3 = 1.7$, $x_4 = 2.0$, $x_5 = 2.3$, $x_6 = 2.8$, $x_7 = 3.2$, $x_8 = 3.5$, $x_9 = 3.7$, $x_{10} = 3.9$, then we have

$$\sum_{i=1}^{10} f(x_i)(y_i - y_{i-1}) = f(1.2)(0.2) + f(1.3)(0.3) + f(1.7)(0.3) + f(2.0)(0.3)$$

$$+ f(2.3)(0.4) + (2.8)(0.5) + f(3.2)(0.4) + f(3.5)(0.2) +$$

$$+ f(3.7)(0.2) + f(3.9)(0.2)$$

$= 15.12.$ #

3.5 DEFINITION. Let F and f be two real-valued functions defined on $[a,b]$ such that $F'(x) = f(x)$ for all $x \in [a,b]$. Then we say that F is the ***antiderivative*** of f in $[a,b]$.

REMARK. One class of Riemann integrable functions is the class of functions which are derivatives of some functions. In fact, *integration as inverse to differentiation has been always emphasized in calculus courses.* The inverse process cannot always be performed. However, in cases where it can be done, computation is made much easier.

3.6 THEOREM. *The following formulas can be shown by obtaining the derivative of the right-hand side using the definition (see also the preceding remark).*

(a) $\int dx = x + C$

(b) $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1$

(c) $\int \dfrac{1}{x} dx = \ln |x| + C$

(d) $\int e^x dx = e^x + C$

(e) $\int \sin x \, dx = \cos x + C$

(f) $\int \cos x \, dx = -\sin x + C$

(g) $\int \sec^2 x \, dx = \tan x + C$

(h) $\int \csc^2 x \, dx = -\cot x + C$

(i) $\int \sec x \tan x \, dx = \sec x + C,$

(j) $\int \csc x \cot x \, dx = -\csc x + C,$

(k) $\int \sec x \, dx = \ln |\sec x + \tan x| + C,$

(i) $\int \csc x \, dx = \ln |\csc x - \cot x| + C,$

(m) $\int \tan x \, dx = \ln |\cos x| + C,$

(n) $\int \cot x \, dx = -\ln |\sin x| + C,$

(o) $\int a^x dx = \dfrac{a^x}{\ln a} + C, \; a > 0,$

(p) $\int \dfrac{1}{x^2+1} dx = \arctan x + C,$

(q) $\int \dfrac{1}{\sqrt{1-x^2}} dx = \arcsin x + C,$

(r) $\int \dfrac{1}{x\sqrt{x^2-1}} dx = \text{arcsec}\, x + C.$

3.7 THEOREM. *If F and G are both antiderivatives of f then G(x) = F(x) + C where C is any constant.*

Proof. Differentiate $G - F$ and use the fact that the antiderivative of the zero function is a constant function. #

This theorem says that any antiderivative of f can be obtained from F by choosing the constant C as needed.

From Definition 3.3, we can see immediate applications of the definite integral which will be discussed later.

3.8 THEOREM. *If F is the antiderivative of f, then* $\int_a^b f(x)dx = F(b) - F(a)$, *i.e., the integral of f over* $[a,b]$ *is* $F(b) - F(a)$.

Proof. F is continuous for each $x \in [a,b]$ since F is the antiderivative of f. By the Mean Value Theorem, $F(y_i) - F(y_{i-1}) = f(x_i)(y_i - y_{i-1})$ for some x_i in $[y_{i-1}, y_i]$. The conclusion follows from this observation. #

3.9 THEOREM. *If* g *is a differentiable function on* $[a,b]$, *then*

$$\int_a^b [g(x)]^n g'(x)dx = \frac{g(x)^{n+1}}{n+1} + C.$$

3.10 DEFINITION. If f is integrable on $[a,b]$, then we define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

Simple properties of the integral. Viewing the definite integral as an area gives a meaningful interpretation of the following theorems.

3.11 THEOREM. *If* f *is integrable on* $[a,b]$, *then*

$$\int_c^c f(x)dx = 0$$

for all c *in* $[a,b]$.

3.12 THEOREM. *If* f *is integrable on the intervals* $[a,c]$, *and* $[c,b]$ *where* $a < c < b$, *then f is integrable on* $[a,b]$ *and*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

3.12.1 EXAMPLE. $\int_0^1 x^2 dx = \int_0^{1/2} x^2 dx + \int_{1/2}^1 x^2 dx$

$$= \frac{1}{3}x^3\Big|_{x=0}^{1/2} + \frac{1}{3}x^3\Big|_{x=1/2}^{1}$$

$$= \frac{1}{3}(\frac{1}{8} - 0) + \frac{1}{3}(1 - \frac{1}{8}) = \frac{1}{3}. \qquad \#$$

3.12.2 COROLLARY. *If f is integrable on the intervals $[a,b]$ and $[c,b]$, where $a < c < b$, then f is also integrable on the interval $[a,c]$ and*

$$\int_a^c f(x)dx = \int_a^b f(x)dx - \int_c^b f(x)dx .$$

3.13 THEOREM. *If f is integrable on $[a,b]$ and k is any constant, then kf is integrable on $[a,b]$ and*

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx .$$

3.14 THEOREM. *If f and g are integrable on $[a,b]$, then $f+g$ is integrable on $[a,b]$ and*

$$\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)dx + \int_a^b g(x)dx .$$

3.14.1 EXAMPLE. $\int_0^1 (x^2 + 2x)dx = \int_0^1 x^2 dx + \int_0^1 2x dx$

$$= \frac{1}{3}x^3 + x^2\Big|_{x=0}^{1} = \frac{4}{3}. \qquad \#$$

3.15 THEOREM. *If f and g are integrable on* $[a,b]$ *and if* $f(x) \geq g(x)$, *then*

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

3.16 THEOREM. *If f is integrable on* $[a,b]$ *and* $m \leq f(x) \leq M$ *for all x in* $[a,b]$, *then*

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

3.17 THEOREM. *If the function* f *is continuous on the interval* $[a,b]$, *there exists a number* $c \in [a,b]$ *such that*

$$\int_a^b f(x)dx = f(c)(b-a).$$

3.17.1 EXAMPLE. Let f be the function $f(x) = x^2$, *with* $a = 0$ *and* $b = 2$. Then

$$\int_0^2 x^2 dx = \frac{8}{3}, \quad c = \frac{2\sqrt{3}}{3} \quad \text{and} \quad f(c)(b-a) = (\frac{4}{3})(2-0) = \frac{8}{3}. \quad \#$$

The succeeding two theorems show the link between the integral and the derivative. Their names suggest the importance of the ideas presented.

3.18 THEOREM. (FIRST FUNDAMENTAL THEOREM OF CALCULUS). *Let the function* f *be continuous on the interval* $[a,b]$ *and let* x *be any number in* $[a,b]$. *If* F *is the function defined by*

$$F(x) = \int_a^x f(t)dt,$$

then $F'(x) = f(x)$ *for all* $x \in [a,b]$.

3.19 THEOREM. (SECOND FUNDAMENTAL THEOREM OF CALCULUS). *Let the function f be continuous on the interval $[a,b]$ and let g be a function such that $g'(x) = f(x)$ for all $x \in [a,b]$. Then*

$$\int_a^b f(x)dx = g(b) - g(a).$$

3.20 DEFINITION. A partition $D : a = y_0 < y_1 < \cdots < y_n = b$ of $[a,b]$ is said to be ***regular*** if the subintervals $[y_i, y_{i-1}]$ are of equal lengths for all $i = 1, 2, \ldots, n$.

3.21 THEOREM. *If the function f is continuous on the closed interval $[a,b]$ and the numbers $a = x_0 < x_1 < \cdots < x_n = b$ form a regular partition of $[a,b]$, then*

$$\int_a^b f(x)dx = \frac{(b-a)}{2n}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Integration techniques. We shall look at some of the most useful techniques for evaluating integrals.

(1) INTEGRATION BY PARTS. Under certain integrability conditions, e.g., continuity of the derivatives f' and g', the following formula holds:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

EXAMPLE. Integrate $\int x^3 e^{x^2} dx$.

Solution. Let $f(x) = x^2$ and $g'(x)dx = x e^{x^2} dx$, then $f'(x)dx = 2xdx$ and $g(x) = \frac{1}{2}e^{x^2}$. Hence,

$$\int x^3 e^{x^2} dx = x^2 \frac{1}{2} e^{x^2} - \int \frac{1}{2} e^{x^2} 2xdx$$

$$= x^2 \frac{1}{2} e^{x^2} - \int e^{x^2} xdx$$

$$= x^2 \frac{1}{2} e^{x^2} - \frac{1}{2} e^{x^2} + C. \quad \#$$

EXAMPLE. Integrate $\int x^2 e^x dx$.

Solution. Let $f(x) = x^2$ and $g'(x)dx = e^x dx$, then $f'(x)dx = 2xdx$ and $g(x) = e^x$. Hence,

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx + C'.$$

Now let $\bar{f}(x) = x$ and $\bar{g}'(x)dx = e^x dx$, then $\bar{f}'(x)dx = dx$ and $\bar{g}(x) = e^x$. Hence,

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C'' \quad \text{and}$$

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx + C'$$

$$= x^2 e^x - 2(xe^x - e^x + C'') + C'$$

$$= x^2 e^x - 2xe^x - 2e^x + C, \text{ where } C = C' + 2C''. \quad \#$$

EXAMPLE. Integrate $\int e^x \cos x \, dx$.

Solution. Let $f(x) = e^x$ and $g'(x)dx = \cos x \, dx$, then $f'(x)dx = e^x$ and $g(x) = \sin x$. Thus,

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x \, dx + C'.$$

Now let $\bar{f}(x) = e^x$ and $\bar{g}'(x)dx = \sin x \, dx$ then $\bar{f}'(x)dx = e^x dx$ and $\bar{g}(x) = -\cos x$. Hence,

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx + C'' \quad \text{and}$$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx + C'$$

$$= e^x \sin x - (-e^x \cos x + \int e^x \cos x \, dx + C'') + C'$$

Therefore,

$$2\int e^x \cos x \, dx = e^x \sin x \; + \; e^x \cos x + C \text{ , where } C \; = \; C' - C''. \quad \#$$

(2) TRIGONOMETRIC SUBSTITUTIONS. Integration by trigonometric substitution is based on the Pythagorean identities. There are three types:

(*i*) $\int \frac{du}{\sqrt{a^2 - u^2}}$, $a > 0$. Let $u \; = \; a \sin \theta$. Then $du = a \cos \theta \, d\theta$ and $\sqrt{a^2 - u^2} \; = \; a \cos \theta$.

EXAMPLE. $$\int \frac{u^2 du}{\sqrt{9 - u^2}} = \int \frac{(3\sin \theta)^2 \cos \theta \, d\theta}{3\cos \theta}$$

$$= 9\int \sin^2\theta \, d\theta$$

$$= \frac{9}{2}(\theta - \frac{1}{2}\sin 2\theta) + C$$

$$= \frac{9}{2}(\arcsin (\frac{u}{3}) - \sqrt{9 - u^2}\,) + C. \quad \#$$

(*ii*) $\int \frac{du}{\sqrt{a^2 + u^2}}$, $a > 0$. Let $u = a \tan \theta$. Then $du \; = a \sec^2 \theta \, d\theta$ and $\sqrt{a^2 + u^2} = a \sec \theta$.

EXAMPLE. $$\int \frac{du}{\sqrt{4 + u^2}} = \int \frac{2\sec^2\theta \, d\theta}{2\sec \theta}$$

$$= \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4+u^2}}{2} + \frac{u}{2}\right| + C. \quad \#$$

(iii) $\int \frac{du}{\sqrt{u^2 - a^2}}$, $a > 0$. Let $u = a\sec\theta$. Then $du = \sec\theta\tan\theta\, d\theta$ and $\sqrt{u^2 - a^2} = a\tan\theta$.

EXAMPLE. $\int \frac{du}{\sqrt{u^2 - 25}} = \int \frac{5\sec\theta\tan\theta\, d\theta}{5\tan\theta}$

$$= \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{u}{5} + \frac{\sqrt{u^2 - 25}}{5}\right| + C. \quad \#$$

(3) PARTIAL FRACTIONS. This method is based on the fact that a rational function of the form $\frac{P(u)}{Q(u)}$ can be expressed as a sum of simple partial fractions.

For instance, suppose $Q(u)$ is a product of nonrepeating linear factors. Let $Q(u) = L_1(u)L_2(u)$, where $L_1(u)$ and $L_2(u)$ are linear. Then for some $A, B \in \mathbf{R}$,

$$\int \frac{P(u)}{Q(u)}\,du = \int \left[\frac{A}{L_1(u)} + \frac{B}{L_2(u)} \right] du\,.$$

EXAMPLE. $$\int \frac{5u-3}{(u+1)(u-3)}\,du = \int \left[\frac{A}{(u+1)} + \frac{B}{(u-3)} \right] du$$

$$= \int \frac{A(u-3)+B(u+1)}{(u+1)(u-3)}\,du$$

$$= \int \frac{(A+B)u \;+\; (B-3A)}{(u+1)(u-3)}\,du$$

$$= \int \left[\frac{2}{(u+1)} + \frac{3}{(u-3)} \right] du \qquad (A+B=5, \text{ and } B-3A=-3)$$

$$= 2\ln|u+1| + 3\ln|u-3| + C\,.$$

The last integral is obtained by equating the coefficients of the numerators in the first integral and the fourth integral. #

REMARKS. (1) If $Q(u)$ is a product of nonrepeating linear factors, i.e., if $Q(u) = L_1(u)L_2(u)$, where $L_1(u) = L_2(u)$, then, there exist real numbers A and B such that,

$$\int \frac{P(u)}{Q(u)}\,du = \int \left[\frac{A}{L_1(u)} + \frac{B}{L_1(u)^2} \right] du\,.$$

(2) If $Q(u)$ has a quadratic factor, say, $Q(u) = L(u)M(u)$, where $L(u)$ is linear and $M(u)$ is quadratic, then, there exist real numbers A, B and C such that,

$$\int \frac{P(u)}{Q(u)}\,du = \int \left[\frac{A}{L(u)} + \frac{Bu+C}{M(u)} \right] du\,.$$

Applications. We shall now consider some applications of the integral.

3.22 DEFINITION. Let the function f be continuous on $[a,b]$ and $f(x) \geq 0$ for all x in $[a,b]$. Let R be the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$. Then the ***area*** A of the region R is given by

$$A = \int_a^b f(x)dx .$$

3.22.1 EXAMPLE. Take f to be the line passing through the origin with slope 1. Let $a = 1$ and $b = 3$. Then

$$A = \int_1^3 xdx = \frac{1}{2}x^2\Big|_{x=1}^{3} = \frac{1}{2}(9-1) = 4 . \quad \#$$

3.23 DEFINITION. Let S be a solid such that S lies between planes drawn perpendicular to the x-axis at a and b. If the area of the plane section of S drawn perpendicular to the x-axis at x is given by $A(x)$, where A is continuous on $[a,b]$, then the ***volume*** of S, V, is given by

$$V = \int_a^b A(x)dx .$$

3.23.1 EXAMPLE. Let S be a pyramid with a square base whose plane is perpendicular to the x-axis at the origin. If the side of the base is 10 units and the height is 6 units, what is the volume of S?

Solution. The area of the plane section of S drawn perpendicular to the x-axis at x units from the origin is a square. Using ratio and proportion on the right triangle formed by the altitude and the base we have

$$A(x) = \left(\frac{30-5x}{3}\right)^2$$

$$V = \int_0^6 A(x)dx = \int_0^6 \left(\frac{30-5x}{3}\right)^2 dx$$

$$= \int_0^6 \frac{1}{9}(900 - 300x + 25x^2)dx$$

$$= \frac{1}{9}(900x - 150x^2 + \frac{25}{3}x^3)\Big|_{x=0}^{6} = 200 \text{ cubic units. } \#$$

3.24 THEOREM (DISC METHOD) *Let the function* f *be continuous on the closed interval* $[a,b]$ *and assume* $f(x) \geq 0$ *for all* x *in* $[a,b]$. *If* S *is the solid of revolution obtained by revolving about the x-axis the region bounded by the curve* $y = f(x)$, *the x-axis, and the lines* $x = a$ *and* $x = b$, *and if* V *is the volume of* S, *then*

$$V = \pi \int_a^b [f(x)]^2 dx .$$

3.24.1 EXAMPLE. The region between the curve $y = \sqrt{x}$, $0 < x < 4$ and the x-axis is revolved about the x-axis to generate a solid. Find the volume of the solid.

$$V = \pi \int_0^4 [\sqrt{x}]^2 dx = 8\pi. \ \#$$

3.25 THEOREM. (WASHER METHOD) *Let the functions* f *and* g *be continuous on the closed interval* $[a,b]$ *and assume* $f(x) \geq 0$ *for all* x *in* $[a,b]$. *If* S *is the solid of revolution obtained by revolving about the x-axis the region bounded by the curve* $y = f(x)$ *and* $y = g(x)$ *and the lines* $x = a$ *and* $x = b$, *then the volume,* V, *of* S *is given by*

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2)dx .$$

3.25.1 EXAMPLE. The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

$$V = \pi \int_{-2}^{1} ([-x+3]^2 - [x^2+1]^2)dx$$

$$= \pi \left(8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_{x=-2}^{1}$$

$$= \frac{117\pi}{5}. \quad \#$$

3.26 THEOREM (SHELL METHOD). *Let the function f be continuous on the closed interval $[a,b]$ where $a \geq 0$ and assume $f(x) \geq 0$ for all x in $[a,b]$. If S is the solid of revolution obtained by revolving about the x-axis the region bounded by the curve $y = f(x)$, the y-axis, and the lines $x = a$ and $x = b$, and if V is the volume of S, then*

$$V = 2\pi \int_a^b xf(x)dx.$$

3.26.1 EXAMPLE. The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution. Using the preceding formula for the volume, we have

$$V = 2\pi \int_0^4 x\sqrt{x}dx = 2\pi(\frac{2}{5}x^{5/2})\Big|_{x=0}^{4} = \frac{128\pi}{5}. \quad \#$$

3.27 DEFINITION. Suppose the function f is continuous on the closed interval $[a,b]$. Further suppose that there exists a number L having the following property: For any $\varepsilon > 0$ there is $\delta > 0$ such that for every partition D of the interval $[a,b]$ it is true that if $\|D\| < \delta$, then we have

$$\left|\sum_i \left|P_{i-1}P_i\right| - L\right| < \varepsilon.$$

Then L is called the ***arclength*** of the curve $y = f(x)$ from the point $A(a,f(a))$ to the point $B(b,f(b))$.

3.28 THEOREM. *If the function f and its derivative f' are continuous on $[a,b]$, then the length of the curve $y = f(x)$ from the point $(a, f(a))$ to the point $(b,f(b))$ is given by*

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx .$$

3.28.1 EXAMPLE. Find the length of the arc of the curve $y = x^{2/3}$ from the point $(1,1)$ to the point $(8,4)$.

Solution. The arclength is given by

$$L = \int_1^8 \sqrt{1+[\frac{2}{3}x^{-1/3})]^2}\,dx = \frac{1}{3}\int_1^8 \frac{\sqrt{9x^{2/3}+4}}{x^{1/3}}dx .$$

Let $u = 9x^{2/3} + 4$. Then $du = 6x^{-1/3}dx$. Hence,

$$L = \frac{1}{18}\int_{13}^{40} u^{1/2}du = \frac{1}{18}\left(\frac{2}{3}u^{3/2}\right)\Bigg|_{u=13}^{40} = \frac{1}{27}(40^{3/2} - 13^{3/2}).\quad \#$$

3.29 DEFINITION. The ***moment of mass*** of a particle of mass m located x units away from the origin is given by mx.

3.30 DEFINITION. A rod of length L meters has its left endpoint at the origin. If $\delta(x)$ kilograms per meter is the *linear density* at a point x meters from the origin, where δ is continuous on $[0,L]$, then the ***total mass*** of the rod is M kilograms, where

$$M = \int_0^L \delta(x)dx .$$

3.31 DEFINITION. A rod of length L meters has its left endpoint at the origin. If $\delta(x)$ kilograms per meter is the linear density at a point x meters from the origin, where δ is continuous on $[0, L]$, then the ***moment of mass*** of the rod with respect to the origin is given by

$$M_0 = \int_0^L x\delta(x)dx .$$

The ***center of mass*** is at

$$\bar{x} = \frac{M_0}{M} .$$

3.31.1 EXAMPLE. A 10-meter-long rod thickens from left to right so that its density is given by $\delta(x) = 1 + \frac{x}{10}$ kg./m.. Find the rods center of mass.

Solution. The mass M and the moment M_0 of the rod is given by the equation

$$M = \int_0^{10} \left(1 + \frac{x}{10}\right)dx = \left(x + \frac{x^2}{20}\right)\Bigg|_{x=0}^{10} = 15 \text{ kg.},$$

and the equation

$$M_0 = \int_0^{10} x\left(1 + \frac{x}{10}\right)dx = \left(\frac{x^2}{2} + \frac{x^3}{30}\right)\Bigg|_{x=0}^{10} = \frac{250}{3}\text{kg.}-\text{m.}.$$

Hence, the center of mass is at

$$\bar{x} = \frac{M_0}{M} = \frac{\frac{250}{3}\text{kg.-m.}}{15 \text{ kg.}} = \frac{50}{9}. \quad \#$$

3.32 DEFINITION. Let L be a homogeneous lamina whose constant area density is k kilograms per square meter and which is bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$. The function f is continuous on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$. If M_y kilogram-meters is the ***moment of mass of the lamina*** L ***with respect to the y-axis***, then

$$M_y = k\int_a^b xf(x)dx$$

If M_x kilogram-meters is the ***moment of mass of the lamina*** L ***with respect to the x-axis***, then

$$M_x = \frac{1}{2}k\int_a^b [f(x)]^2\,dx .$$

If M kilograms is the ***total mass of the lamina*** L, then

$$M = k\int_a^b f(x)dx .$$

If $(\bar{x}, \bar{y})$ is the ***center of mass of the lamina*** L, then

$$\bar{x} = \frac{\frac{1}{2}\int_a^b [f(x)]^2\,dx}{\int_a^b f(x)dx} = \frac{M_x}{M}, \quad \text{and} \quad \bar{y} = \frac{\frac{1}{2}\int_a^b xf(x)dx}{\int_a^b f(x)dx} = \frac{M_y}{M}.$$

3.32.1 EXAMPLE. Find the center of mass of the region bounded by $f(x) = 2\sqrt{x}$, $x = 4$ and the x-axis.

Solution. From the preceding formulas

$$M = k\int_0^4 2\sqrt{x}dx = \frac{4}{3}kx^{3/2}\Big|_0^4 = \frac{32}{3}k.$$

$$M_x = \frac{1}{2}k\int_0^4 [2\sqrt{x}]^2\,dx = \frac{1}{2}k\int_0^4 4x\,dx = \frac{1}{2}k(2x^2)\Big|_{x=0}^4 = -32k.$$

$$M_y = k\int_0^4 x(2\sqrt{x})dx = 2k\int_0^4 x^{3/2}dx = \frac{4k}{5}x^{5/2}\Big|_{x=0}^{4} = 16k$$

$$\bar{x} = \frac{M_x}{M} = \frac{16k}{\frac{32}{3}k} = \frac{3}{2} \text{ and } \bar{y} = \frac{M_y}{M} = \frac{\frac{128}{5}k}{\frac{32}{3}k} = \frac{12}{5}. \quad \#$$

4.33 DEFINITION. Let the function F be continuous on the closed interval $[a,b]$ and $F(x)$ be the force acting on an object at the point x on the x-axis. Then if W is the ***work*** done by the force as the object moves from a to b,

$$W = \int_a^b F(x)dx .$$

3.33.1 EXAMPLE. A leaky bucket is lifted to a height of 20 feet. If the bucket starts with 16 lbs. of water and leaks it at a constant rate emptying the water as it reaches the full height, how much work was done in lifting the water?

Solution. The force used in lifting the bucket diminishes with the height traveled by the bucket and is proportional to the height. (We shall disregard the weight of the bucket.) This force is given by

$$F(x) = 16\left(\frac{20-x}{20}\right) = \frac{4}{5}(20-x), \text{ so}$$

$$W = \int_0^{20}\frac{4}{5}(20-x)dx = \frac{4}{5}(20x-\frac{1}{2}x^2)\Big|_{x=0}^{20} = 160 \text{ lbs.. } \quad \#$$

3.34 DEFINITION. Suppose that a flat plate is submerged vertically in a liquid for which a measure of its *mass density* is ρ. The length of the plate at a depth of x units below the surface of the liquid is $f(x)$ units, where f is continuous on the closed interval $[a,b]$ and $f(x) \geq 0$ on $[a,b]$. Then if F is the measure of the force caused by ***liquid pressure*** on the plate,

$F = \int_a^b \rho g x f(x) dx$, where g is the gravitational constant.

3.34.1 EXAMPLE. A trough having a trapezoidal cross section is full of water. If the trapezoid is 3 feet wide at the top, 2 feet wide at the bottom and 2 feet deep. Find the total force owing the water pressure on one end of the trough.

Solution. In this problem $f(x) = \frac{3}{2} - \frac{1}{4}x$, which is obtained from the equation of the line which form one side of the trapezoid. The force F is

$$F = 2\int_a^b \rho g x\left(\frac{3}{2} - \frac{1}{4}x\right)dx$$

$$= 2\ \rho g\left(\frac{3}{4}x^2 - \frac{1}{12}x^3\right)\Big|_{x=0}^{2}$$

$$= \rho g\frac{14}{3} = \frac{14}{3}(62.5\text{ lbs.}).\ \#$$

3.35 DEFINITION. Let the curve C have the parametric equation $x = f(t)$ and $y = g(t)$. Suppose there exists a number L having the property: For any $\varepsilon > 0$ there is a $\delta > 0$ such that for every partition D of the interval $[a,b]$ for which $\|D\| < \delta$, then

$$\left|\sum_{i=1}^{n} \overline{P_{i-1}P_i} - L\right| < \varepsilon.$$

Then we write

$$L = \lim_{\|D\|\to 0} \sum_{i=1}^{n} \overline{P_{i-1}P_i}\ ,$$

and L is called the ***length of arc*** of the curve C from the point $(f(a),g(a))$ to the point $(f(b),g(b))$.

3.36 THEOREM. *If the function F and G are continuous on the closed interval $[a,b]$ then the function $\sqrt{F^2+G^2}$ is also continuous on the closed interval $[a,b]$ and if D is a partition $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a,b]$ and z_i and w_i are any numbers in (t_{i-1}, t_i), then*

$$\lim_{\|D\|\to 0} \sum_{i=1}^{n} \sqrt{[F(z_i)]^2 + [G(w_i)]^2}\, \Delta_i t = \int_a^b \sqrt{[F(t)]^2 + [G(t)]^2}\, dt .$$

3.37 THEOREM. *Let the curve C have parametric equations $x = f(t)$ and $y = g(t)$, and suppose that f' and g' are continuous on the closed interval $[a,b]$. Then if L units is the length of arc of the curve C from the point $(f(a),g(a))$ to the point $(f(b),g(b))$, then*

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} .$$

3.38 DEFINITION. Let f be a function defined on a closed rectangular region R. The number L is said to be the ***limit of sums*** of the form

$$\sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta_i A$$

if L satisfies the property that for any $\varepsilon > 0$ there is a $\delta > 0$ such that for every partition D of R for which $\|D\| < \delta$ and for all possible choices of the point (ξ_i, η_i) in the ith rectangle $i = 1, 2, \ldots, n$

$$\left| \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta_i A - L \right| < \varepsilon .$$

If such L exists, we write

$$\lim_{\|D\|\to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta_i A = \iint_R f(x,y)\,dA$$

3.39 THEOREM. *Suppose f is a function of two variables that is continuous on a closed rectangular region R, in the xy-plane, and*

$f(x,y) \geq 0$ for all (x,y) in R. If V cubic units is the volume of the solid S having the region R as its base and having an altitude of $f(x,y)$ units at the point (x,y) in R, then

$$V = \lim_{\|D\|\to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta_i A = \iint_R f(x,y)\,dA$$

It would be difficult if not impossible to find the double integral of a function from the definition. Fortunately, a lot of the double integrals we would be interested in can be found using techniques we have learned in single integration. The following theorem shows how this can be done.

3.40 THEOREM (FUBINI'S THEOREM). *If f is integrable on a rectangle $R = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}$ and suppose that for each value y in $[c,d]$ the integral $F(y) = \int_a^b f(x,y)dx$ exists. Then F is integrable on $[c,d]$ and*

$$\iint_R f(x,y)dA = \int_c^d F(y)dy = \int_c^d \left[\int_a^b f(x,y)dx \right] dy$$

3.40.1 COROLLARY. *Let A be the region given by $A = \{ (x,y) : p(y) \leq x \leq q(y),\ c \leq y \leq d \}$ where p and q are continuous functions on $[c,d]$ with values in $[a,b]$. If f is continuous and real valued on A, then f is integrable on A and*

$$\iint_A f(x,y)dA = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)dx \right] dy.$$

Items 38 to 40.1 can be altered using $f(\xi_i, \eta_i, \zeta_i)$ and $\iiint f(x,y,z)dV$ to give properties for the triple integral.

A lot of applications use integrals evaluated on a domain which is a path rather than of regions with areas. A discussion on these type of integrals are available in most calculus books.

Let us end with a note that what have been discussed here are just the introductory part of applications. We hope that this training will serve as an appetizer for your mathematical feast.

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