SEQUENCES AND SERIES

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The limit concept is the essence of analysis. Indeed, many important analytical constructs are defined using various limit processes. The derivative and the integral are the two main examples. In calculus, there is without doubt a need for a thorough understanding of limits. "When students fail to understand this idea," Gaughan writes in [3; p.59], "their study of calculus becomes a drudgery of juggling formulas." To this we add: when teachers fail to understand limits, their teaching of calculus is not much different from a 'cooking' demonstration.

This lecture is a basic introduction to the theory of limits of sequences. It is intended for the teacher/participants of the *First Mindanao Mathematics Teachers' Training-Seminar*. The material should cover the minimum needed by these teachers. There is a serious attempt to present it rigorously and, in most cases, detailed proofs are included. After all, a good grasp of the fundamentals is almost synonymous with the ability to prove the basic assertions. Here we introduce a new approach to convergence, which appears to be interesting and attractive.

The lecture is divided into two parts: first, we cover the rudiments of limit theory, then we use it to clarify the meaning of an infinite sum.

Convergence. A ***sequence*** of real numbers (s_n) is a real-valued function on the set **N** of natural numbers or positive integers. A sequence is usually described by a formula such as $s_n = 3/n$ or a list such as

$$(3, \tfrac{3}{2}, 1, \tfrac{3}{4}, \tfrac{3}{5}, \ldots).$$

As functions, sequences are very simple mathematical objects. Because of this, they make a suitable vehicle for a leisurely introduction to the theory of limits.

To show that a sequence (s_n) of real numbers ***converges*** to a real number s, the old procedure, which goes way back to the time of Augustin

Louis Cauchy (1789-1867), consists of two steps: (1) to find an integer N corresponding to a given $\varepsilon > 0$, and (2) to show that if the index $n \geq N$, then we must have $|s_n - s| < \varepsilon$.

Each of these steps constitutes almost half of the whole procedure and it usually takes some time before the average student can master the intricacies of both. However, only step (2) is essential. One can view this problem from a better perspective if he bears in mind that:

> *A sequence* (s_n) *converges to a real number* s *if the difference* $|s_n - s|$ *eventually becomes very, very small.*

More precisely, this means that for any specific $\varepsilon > 0$, no matter how small, the inequality $|s_n - s| < \varepsilon$ must eventually hold. In particular, we must guarantee this when n is large. This exactly is the content of the definition of convergence.

4.0 DEFINITION. A statement or a condition $C(n)$ ***holds for large*** n, if there is a positive integer N such that $C(n)$ holds for all $n \geq N$.

4.1 DEFINITION. We say that a sequence (s_n) of real numbers ***converges to a limit*** s, if for each $\varepsilon > 0$, the inequality

$|s_n - s| < \varepsilon$ holds for large n.

In this case we shall write $\lim s_n = s$ or $\lim_{n\to\infty} s_n = s$, and we shall say that the sequence (s_n) is a ***convergent*** sequence; otherwise, we shall say that the sequence (s_n) is ***divergent***.

In this definition, all our attention is focused on the difference $|s_n - s|$. In order to prove that the sequence (s_n) converges to the real number s, we must show that this difference can be made small.

Definition 4.1, which is essentially due to Redheffer [1], is based on a mode of expression that has long been a part of the mathematical jargon. The expression '*for large n*' is often used in the literature, although it is never defined explicitly; it is simply taken for granted that its meaning is clear and it needs no further explanation. Consider for instance the following statement:

$n^2 - 7n - 30 \geq 0$ for large n.

Based on Definition 4.0, this statement is true. For if $n \geq 37$, then

$$n^2 \geq 37n = 7n + 30n \geq 7n + 30.$$

Hence, $n^2 - 7n - 30 \geq 0$ for all $n \geq 37$. #

Here is another illustration of Definition 4.0.

4.2 LEMMA. *If a, b, and p are positive real numbers, then for any real number k the inequality* $\dfrac{a}{(n-k)^p} < b$ *holds for large n.*

Proof. According to the Archimedean property of real numbers, there is a positive integer N, large enough, so that

$$Nb^{1/p} > a^{1/p} + kb^{1/p}.$$

(Almost obvious, since the set of positive integers is *not bounded.*) Hence,

$$(n-k)\ b^{1/p} > a^{1/p} \text{ for all } n \geq N.$$

Hence, by Definition 4.0,

$$(n-k)\ b^{1/p} > a^{1/p} \text{ for large } n.$$

Therefore, $b(n-k)^p > a$ for large n, and the conclusion follows. #

The next lemma is a vital cornerstone in our approach.

4.3 THE CONJUNCTION LEMMA. *Let $C_1(n)$ and $C_2(n)$ be two conditions on n, where $n \in$* **N**. *If $C_1(n)$ holds for large n and $C_2(n)$ holds for large n, then the conjunction $C_1(n)$ & $C_2(n)$ also holds for large n.*

Proof. By hypothesis, suppose that $C_1(n)$ holds for $n \geq N_1$, and $C_2(n)$ holds for $n \geq N_2$, where N_1 and N_2 are positive integers. Let $N = \max\{N_1, N_2\}$. Then both $C_1(n)$ and $C_2(n)$ hold for $n \geq N$. When this happens, the conjunction $C_1(n)$ & $C_2(n)$ also holds. Therefore, $C_1(n)$ & $C_2(n)$ holds for large n. #

We immediately have two useful corollaries. Let (s_n), (t_n), (u_n), and (v_n) be sequences of real numbers.

4.3.1 COROLLARY. *If $s_n < t_n$ holds for large n and $t_n < u_n$ holds for large n, then $s_n < u_n$ holds for large n.*

4.3.2 COROLLARY. *If $s_n < t_n$ holds for large n and $u_n < v_n$ holds for large n, then $s_n + u_n < t_n + v_n$ holds for large n.*

Limit theorems. How simple and effective is our "new" definition of convergence? As typical illustrations of our approach, we shall prove more useful results.

4.4 THEOREM. *If $a, k \in \mathbf{R}$, and $p > 0$, then* $\lim\limits_{n\to\infty} \dfrac{a}{(n-k)^p} = 0.$

Proof. Let $\varepsilon > 0$. It follows from Lemma 4.2, with $b = \varepsilon$, that

$$\left|\frac{a}{(n-k)^p} - 0\right| = \frac{|a|}{(n-k)^p} < \varepsilon \text{ holds for large } n.$$

Hence, by Definition 4.1, the desired conclusion follows. #

4.5 THEOREM. *If $s_n = k$ for large n, then* $\lim\limits_{n\to\infty} s_n = k$.

Proof. The hypothesis is equivalent to the statement $|s_n - k| = 0$ for large n. The desired conclusion follows from Definition 4.1. #

4.5.1 COROLLARY. *The constant sequence (k, k, k, ...) converges to k, i.e.,* $\lim\limits_{n\to\infty} k = k$.

4.6 THEOREM. $\lim\limits_{n\to\infty} s_n = s$ *if and only if* $\lim\limits_{n\to\infty} |s_n - s| = 0$.

The proof is an easy exercise. #

4.7 THEOREM. $\lim\limits_{n\to\infty} r^n = 0$, *if* $|r| < 1$.

Proof. Let $0 < \varepsilon < 1$. Then, clearly, $0 < (-\log \varepsilon) < n(-\log |r|)$ for large n. It follows that $|r^n - 0| = |r|^n < \varepsilon$ for large n. #

We shall now prove some standard limit theorems for sequences.

4.8 THE SQUEEZE THEOREM. *Let* $s \in \mathbf{R}$. *If* $|s_n - s| \leq t_n$ *for large n, and* $\lim_{n\to\infty} t_n = 0$, *then* $\lim_{n\to\infty} s_n = s$.

Proof. Let $\varepsilon > 0$. By hypothesis, the inequality

$|s_n - s| \leq t_n$ holds for large n.

By the second hypothesis and by Definition 4.1,

$|t_n - 0| < \varepsilon$ holds for large n.

Thus, by Corollary 4.3.1, the relation

$|s_n - s| \leq t_n = |t_n - 0| < \varepsilon$ also holds for large n.

Therefore, by Definition 4.1, $\lim s_n = s$. #

4.8.1 EXAMPLE. Show that $\displaystyle\lim_{n \to \infty} \frac{\sin(n\pi/2)}{n} = 0.$

Solution. The conclusion follows from the Squeeze Theorem, since

$$\left|\frac{\sin(n\pi/2)}{n}\right| \leq \frac{|\sin(n\pi/2)|}{n} \leq \frac{1}{n} \text{ for all } n. \quad \#$$

4.8.2 EXAMPLE. Prove that

$$\lim_{n\to\infty} \frac{n^2 + 3n - 5}{2n^3 - 5n^2 + n - 4} = 0.$$

Solution. First, we suppose that n is *large enough so that both the numerator and denominator of the given fraction are positive.* (Hence, we can do away with the absolute value signs.) Now observe that for all n,

$$n^2 + 3n - 5 \le n^2 + 3n^2 = 4n^2, \text{ and}$$

$$2n^3 - 5n^2 + n - 4 \ge n^3 - 5n^2 - 4n^2 = n^3 - 9n^2 = (n-9)n^2.$$

(To find an upper estimate, we either increase the negative terms to 0, or increase the powers of the positive terms. To find a lower estimate, we either decrease the coefficients of the positive terms to 1 or 0, or increase the powers of the negative terms.) Hence, it follows that, for large n, we have the relation

$$\left|\frac{n^2 + 3n - 5}{2n^3 - 5n^2 + n - 4}\right| = \frac{n^2 + 3n - 5}{2n^3 - 5n^2 + n - 4} \le \frac{4}{n-9}.$$

Since $\lim_{n\to\infty} \frac{4}{n-9} = 0$, the result follows from the Squeeze Theorem. #

4.8.3 EXAMPLE. Prove that $\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = 0$.

Solution. This result follows from the Squeeze Theorem and Theorem 4.4 with $a = p = 1/2$ and $k = 0$, since

$$\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\left(\sqrt{n+1} - \sqrt{n}\right) = \frac{1}{\sqrt{n+1} + \sqrt{n}} \le \frac{1}{2\sqrt{n}}. \quad \#$$

4.9 THE UNIQUENESS THEOREM. *If* $\lim s_n = L$ *and* $\lim s_n = L'$, *then we shall have* $L = L'$.

Proof. Observe that $|L - L'| \le |s_n - L| + |s_n - L'|$ for all n. Then, it follows from the hypothesis and Theorem 4.6 that $|L - L'| = 0$. #

A sequence (s_n) is ***bounded*** if there is a real number B such that $|s_n| \le B$ for all n.

4.10 THEOREM. *A convergent sequence is bounded.*

Proof. Let $s = \lim s_n$. Then $|s_n - s| < 1$ for large n. By Definition 4.0, there is a positive integer N such that

$|s_n - s| < 1$ for all $n \geq N$.

Now let $B = \max \{ 1 + |s|, |s_1|, |s_2|, \ldots, |s_N| \}$. Clearly, $|s_n| \leq B$, if $n \leq N$. Moreover, $(1 + |s|) \leq B$. Hence, if $n > N$, then, by the triangle inequality, we shall have

$$|s_n| = |s_n - s + s| \leq |s_n - s| + |s| < 1 + |s| \leq B.$$

Therefore, $|s_n| \leq B$ for all n. #

4.10.1 COROLLARY. *An unbounded sequence diverges.*

Algebra of limits. The next group of theorems yields a powerful and elegant tool for proving that a sequence converges to a given number.

4.11 THEOREM. *If $k \in \mathbf{R}$ and the sequence (s_n) converges to $s \in \mathbf{R}$, then the sequence (ks_n) converges to ks, i.e.,* $\lim_{n\to\infty} ks_n = k \lim_{n\to\infty} s_n$.

Proof. Suppose $k \neq 0$ and let $\varepsilon > 0$. Then, by hypothesis,

$$|s_n - s| < \frac{\varepsilon}{|k|} \text{ holds for large } n.$$

Therefore, for all $k \in \mathbf{R}$, $|ks_n - ks| < \varepsilon$ holds for large n. Hence, by Definition 4.1, (ks_n) converges to ks. #

4.12 THEOREM. *If (s_n) converges to s, and (t_n) converges to t, then the sequence of sums $(s_n + t_n)$ converges to $s + t$, i.e.,*

$$\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n.$$

Proof. For each n, it follows from the triangle inequality that

$$|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t|. \tag{4.12.1}$$

Now let $\varepsilon > 0$. Then, by hypothesis,

$|s_n - s| < \varepsilon/2$ holds for large n, and

$|t_n - t| < \varepsilon/2$ holds for large n.

Hence, by Corollary 4.3.2, $|s_n - s| + |t_n - t| < \varepsilon$ holds for large n. Thus, by Corollary 4.3.1 and by (4.12.1),

$$|(s_n + t_n) - (s + t)| < \varepsilon \text{ holds for large } n. \quad \#$$

4.13 THEOREM. *If* $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$, *and* $s_n \le t_n$ *for large* n, *then we have* $s \le t$.

Proof. Let $\varepsilon > 0$. By hypothesis,

$$|s_n - s| < \varepsilon/2 \text{ for large } n, \text{ and } |t_n - t| < \varepsilon/2 \text{ for large } n.$$

Since $s_n \le t_n$ for large n, we also have

$$s - |s_n - s| \le s_n \le t_n \le |t_n - t| + t \text{ for large } n.$$

Hence, $s - t \le |s_n - s| + |t_n - t| < \varepsilon$ for large n. Since ε is arbitrary, it follows that $s - t \le 0$. Therefore, we have $s \le t$. #

4.13.1 COROLLARY. *If* $s_n \le M$, *for large* n, *and* $\lim_{n\to\infty} s_n = L$, *then we have* $L \le M$.

4.14 THEOREM. *If* (s_n) *is bounded for large n and* (t_n) *converges to* 0, *then the sequence* $(s_n t_n)$ *converges to* 0.

Proof. Since there is a $B \in \mathbf{R}$ such that $|s_n| \le B$ for large n, then

$$|s_n t_n - 0| = |s_n t_n| \le B|t_n| \text{ for large } n.$$

Hence, by Theorems 4.8, and 4.11, we have $\lim_{n\to\infty} s_n t_n = 0$. #

4.15 THEOREM. *If* (s_n) *converges to* s, *and* (t_n) *converges to* t, *then the product sequence* $(s_n t_n)$ *converges to* st, *i.e.*,

$$\lim_{n\to\infty} s_n t_n = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right).$$

Proof. First, we observe that for each n, it follows from the triangle inequality that

$$|s_n t_n - st| \le |s_n||t_n - t| + |t||s_n - s|. \tag{4.15.1}$$

The sequence (s_n) is convergent; hence, it is bounded, by Theorem 4.10. Thus, by Theorems 4.6, 4.11, and 4.14, and the hypothesis,

$$\lim_{n\to\infty} \left(|s_n||t_n - t|\right) = 0, \text{ and } \lim_{n\to\infty} \left(|t||s_n - s|\right) = 0.$$

Finally, by Theorem 4.12,

$$\lim_{n\to\infty} |s_n||t_n - t| + |t||s_n - s| = 0.$$

Therefore, in view of (4.15.1) and the Squeeze Theorem, it follows that

$$\lim_{n\to\infty} s_n t_n = st. \quad \#$$

4.16 THEOREM. *If* (s_n) *converges to* s, $s_n \neq 0$ *for all* n, *and* $s \neq 0$, *then the sequence of reciprocals* $(1/s_n)$ *converges to* $1/s$.

Proof. By hypothesis, $|s|/2 > 0$. Also by hypothesis $|s_n - s| < |s|/2$ holds for large n. Hence, it follows from the triangle inequality that

$|s| \le |s - s_n| + |s_n| < |s|/2 + |s_n|$ holds for large n.

Hence, the inequality $0 < |s|/2 < |s_n|$ holds for large n. Now, since each $s_n \neq 0$, then for any $N \ge 1$,

$$\min\left\{|s_n| : n \le N\right\} > 0.$$

Hence, there exists $m > 0$, such that $m \le |s_n|$ for all n. Thus, for each n

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s_n - s|}{|s_n||s|} \le \frac{|s_n - s|}{m|s|}.$$

Using Theorems 4.11 and the Squeeze Theorem, we conclude that

$$\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}. \qquad \#$$

4.17 THEOREM. *Suppose that* (s_n) *converges to* s *and* (t_n) *converges to* t. *If* $s_n \neq 0$ *for all* n *and* $s \neq 0$, *then the quotient sequence* (t_n/s_n) *converges to* t/s.

Proof. By Theorem 4.16, $(1/s_n)$ converges to $(1/s)$. The desired conclusion follows easily from Theorem 4.15. #

4.17.1 EXAMPLE. (*i*) There is a tendency among students to use limit theorems rather carelessly by drawing conclusions without first verifying the hypothesis. This habit sometimes leads to embarrassing results as the following illustration shows: If we take for granted that $\lim\limits_{n\to\infty} n = L$, where L is a nonzero real number, then using Theorem 4.17, we shall have

$$0 = \lim_{n\to\infty} \frac{1}{n} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} n} = \frac{1}{\lim\limits_{n\to\infty} n} = \frac{1}{L}.$$

This leads to the equation 0 = 1, which is absurd. The mistake is the result of using Theorem 4.17 without verifying its hypothesis. #

(*ii*) Show that $\lim\limits_{n\to\infty} \dfrac{4n^3 - 3n}{n^3 + 6} = 4.$

Solution. We shall give a careful algebraic proof of this assertion. First we observe that

$$\lim_{n\to\infty} \frac{4n^3 - 3n}{n^3 + 6} = \lim_{n\to\infty} \frac{4 - \dfrac{3}{n^2}}{1 + \dfrac{6}{n^3}}.$$

Next, we show that $\lim (4 - 3/n^2) = 4$. To this end, we must verify that:

$\lim 4 = 4,$ (Theorem 4.5)

$\lim (-3/n^2) = 0.$ (Theorems 4.4 and 4.11).

Hence, by Theorem 4.12, we have

$\lim (4 - 3/n^2) = 4 + 0 = 4,$

We can show similarly that $\lim (1 + 6/n^3) = 1 + 0 = 1$.

Therefore, by Theorem 4.17, we have

$$\lim_{n\to\infty} \frac{4n^3 - 3n}{n^3 + 6} = \lim_{n\to\infty} \frac{4 - \frac{3}{n^2}}{1 + \frac{6}{n^3}} = \frac{\lim\left(4 - \frac{3}{n^2}\right)}{\lim\left(1 + \frac{6}{n^3}\right)} = \frac{4}{1} = 4.$$

As in Example 4.8.2, the above argument actually proves that

$$\lim_{n\to\infty} \frac{4n^3 - 3n}{n^3 + 6} = 4. \quad \#$$

Existence of limits; subsequences. The preceding theorems address the problem of showing that (*i*) a given sequence (s_n) has a limit and (*ii*) that its limit is a given number s. We now have two methods of attacking such problems. In some cases, however, we are only interested on problem (*i*), i.e., the convergence behavior of the sequence. In this situation, the next theorem is often useful.

4.18 THEOREM. *Suppose that the sequence* (s_n) *is a bounded monotone (nonincreasing or nondecreasing) sequence of real numbers. Then* (s_n) *is convergent.*

Proof. Suppose that (s_n) is nondecreasing. Let s be the least upper bound of (s_n), as guaranteed by the Completeness Axiom. Let $\varepsilon > 0$. Then $(s - \varepsilon)$ is not an upper bound for (s_n). Thus, some $s_N > (s - \varepsilon)$. For $n \geq N$, since $s_n \geq s_N$, we have $|s - s_n| = (s - s_n) < \varepsilon$. #

A ***subsequence*** of a sequence (s_n) is a sequence formed by removing elements or terms of the sequence (s_n). For example, if we delete all odd terms of the sequence (s_n), we obtain the subsequence (s_{2n}). The mother sequence is also considered a subsequence of itself. Note that if the mother sequence converges to a limit L, all its subsequences also converge to the same limit L.

4.18.1 EXAMPLE. Let $a_1 = 1, a_2 = \sqrt{2}, \ldots, a_{n+1} = \sqrt{2a_n}$. Prove that (a_n) is nondecreasing and $\lim_{n\to\infty} a_n = 2$.

Solution. We can easily show by induction that the sequence (a_n) is nondecreasing and $a_n \le 2$ for all n. (Show it.) Suppose now that $\lim a_n = L$. Since (a_{n+1}) is a subsequence of (a_n), we also have $\lim a_{n+1} = L$. But the definition of a_{n+1} implies that $L = \sqrt{2L}$. Therefore, $L = 0$ or 2. Since (a_n) is nondecreasing, $\lim a_n \ge 1$, so we must have $L = 2$. #

4.18.2 EXAMPLE. If $r \ge 1$, prove that $\lim_{n\to\infty} r^{1/n} = 1$.

Solution. Observe that $r^{1/n} \ge 1$ for all n and $(r^{1/n})$ is nonincreasing. (Show it.) Therefore, $\lim r^{1/n}$ exists. Suppose $\lim r^{1/n} = L$. Then $\lim r^{2/n} = L^2$, by Theorem 4.15. But $(r^{1/n})$ is a subsequence of $(r^{2/n})$. Hence, we must also have $\lim r^{1/n} = L^2$. Thus, $L^2 = L$ and hence, either $L = 0$ or $L = 1$. But $L \ne 0$, since $r^{1/n} \ge 1$ for all n. Hence, $L = 1$. #

4.18.3 LEMMA. *Every sequence has a monotone subsequence.*

Proof. (H. Thurston [6]) Assume that a tail of the sequence (s_n) does not contain a greatest member. Without loss, we may suppose that this tail is (s_n) itself. We shall show that (s_n) has a nondecreasing subsequence (s_{n_k}). To this end, suppose $n_1 < n_2 < \cdots < n_k$ have been chosen so that $s_{n_k} > s_n$ for all $n < n_k$. (We may start with $n_1 = 1$.) Since s_{n_k} is not the greatest member of (s_n), we may define

$$n_{k+1} = \min\{\, n : n > n_k \text{ and } s_n > s_{n_k} \,\}$$

Clearly, $n_{k+1} > n_k$ and $s_{n_{k+1}} > s_{n_k}$. Moreover, because n_{k+1} is by definition minimal, if $n_k < n < n_{k+1}$, then we must have $s_n \le s_{n_k}$. Thus, if $n < n_{k+1}$, then $s_n < s_{n_{k+1}}$. The subsequence (s_{n_k}) is nondecreasing.

Now assume that every tail of (s_n) contains a greatest member. We may select the terms of a nonincreasing subsequence of (s_n) as follows:

Choose n_1 so that s_{n_1} is the greatest member of $(s_1, s_2, s_3, \ldots)$. Next, choose n_2 so that s_{n_2} is the greatest member of $(s_{n_1+1}, s_{n_1+2}, s_{n_1+3}, \ldots)$. Then, choose n_3 so that s_{n_3} is the greatest member of the subsequence $(s_{n_2+1}, s_{n_2+2}, s_{n_2+3}, \ldots)$, and so on, ad infinitum. The subsequence (s_{n_k}) is a nonincreasing subsequence of (s_n). #

4.18.4 COROLLARY (THE BOLZANO-WEIERSTRASS THEOREM). *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Suppose (s_n) is a bounded sequence of real numbers. By Lemma 4.18.3, (s_n) has a monotone subsequence. By Theorem 4.18, this monotone subsequence, which is bounded, is convergent. #

Sequence of partial sums; convergence of infinite sums. Adding a finite collection of real numbers is easy. However, the situation is more demanding when the collection is infinite. If this infinite collection is not countable, there are even greater complications. Here we shall confine ourselves to the countable case - a case that can be handled satisfactorily using sequences.

The study of infinite sums has important and fascinating applications in physics, engineering, and in many major areas of mathematics. Because the theory of limits (of sequences) plays a vital role in the study of infinite sums, it is, therefore, not surprising to find this topic included in every standard calculus textbook.

An infinite sum of a sequence (u_k) of real numbers is usually denoted by the symbols

$$\sum_{k=1}^{\infty} u_k, \quad \text{or} \quad \sum u_k, \quad \text{or} \quad u_1 + u_2 + u_3 + \cdots .$$

(The symbol $\sum u_k$ is used when it is clear that the summation index k ranges from 1 to infinity, otherwise the range should be clearly indicated.) Any of the three symbols above is called an ***infinite series***. But these symbols are meaningless, unless there is a definite rule for determining what they represent. Since we know how to determine finite sums, it is but natural to approach the problem of an infinite sum by considering the finite case. The standard technique is to generate finite sums of the form

$$s_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum_{k=1}^{n} u_k .$$

These sums are called ***partial sums*** of the series $\sum_{k=1}^{\infty} u_k$; the sequence (s_n) is called the ***sequence of partial sums*** belonging to the series $\sum_{k=1}^{\infty} u_k$.

4.19 DEFINITION. If $\lim_{n\to\infty} s_n$ exists, then this value is called the ***sum*** of the series. Moreover, we say that the series ***converges*** and we write

$$\sum_{k=1}^{\infty} u_k = \lim_{n\to\infty} s_n .$$

If (s_n) fails to have a limit, then we say that the series ***diverges.*** The reader should note that the symbol $\sum_{k=1}^{\infty} u_k$ stands for two things: *the series itself* and the *sum* of the series. This is an abuse of the notation that can be confusing sometimes; but the use is quite common in the literature.

In effect, the preceding definition says that the following equation holds:

$$\sum_{k=1}^{\infty} u_k = \lim_{n\to\infty} \sum_{k=1}^{n} u_k .$$

In the beginning, many students can hardly reconcile the fact that an infinite series $\sum_{k=1}^{\infty} u_k$ stands for a *sequence* of finite sums and not just a single finite sum like (3 + 2 + 17). Perhaps this is all because of the summation sign Σ, which is present in $\sum_{k=1}^{\infty} u_k$. While it is true that an infinite series and an ordinary finite sum share many common properties, the students should realize the difference between these two entities. They must remember that *an infinite series is a sequence of partial sums.*

4.19.1 EXAMPLE. Find the sum of the following series:

(*i*) $1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots$;

(*ii*) $(1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \cdots$.

Solution. (*i*) At first glance, it appears that the sum in (*i*) is 0. However, a little analysis reveals the contrary. Observe that the *n*th partial sum $s_n = 1$, if *n* is odd, and $s_n = 0$, if *n* is even. Since the sequence (1, 0, 1, 0, 1, 0, ...) does not converge, the series has no sum and is divergent.

(*ii*) Observe that the *n*th partial sum of the series telescopes to the value $s_n = 1 - 1/n$ and, hence, $\lim s_n = 1$. Thus, its sum is 1. #

4.19.2 EXAMPLE. (*i*) Show that the series $\Sigma 1/2^k$ converges. (*ii*) Show in general that if $0 < r < 1$, then

$$\sum_{k=1}^{\infty} r^{k-1} = \frac{1}{1-r}.$$

This series is called a ***geometric series*** with ***common factor*** *r*.

Solution. (*i*) The *n*th partial sum of $\Sigma 1/2^k$ is the geometric series

$$s_n = 1/2 + 1/2^2 + 1/2^3 + \cdots + 1/2^n = \frac{1/2 - 1/2^{n+1}}{1 - 1/2}.$$

Clearly $\lim s_n = 1$. Hence, by definition, $\Sigma 1/2^k = 1$. Part (*ii*) is handled similarly and is left as an exercise. #

4.20 THEOREM. (*i*) *Let* (u_k) *and* (v_k) *be sequences of real numbers and assume that the series* Σu_k *and* Σv_k *both converge. Then* $\Sigma(u_k + v_k)$ *also converges and* $\Sigma(u_k + v_k) = \Sigma u_k + \Sigma v_k$.

(*ii*) *If c is a real number, then* $\Sigma c\, u_k = c \Sigma u_k$.

Proof. (*i*) Let (s_n) and (t_n) be the sequences of partial sums of the series Σu_k and Σv_k, respectively. It is easy to see that $(s_n + t_n)$ is the sequence of partial sums of the series $\Sigma(u_k + v_k)$. Since $\lim (s_n + t_n) = \lim s_n + \lim t_n$, the conclusion follows readily.

Part (*ii*) is proved similarly. #

4.21 THEOREM. *If the series* $\sum u_k$ *converges, then* $\lim_{k\to\infty} u_k = 0$.

Proof. Suppose the series $\sum u_k$ converges to S, i.e., $\lim_{n\to\infty} s_n = S$. Clearly, we also have $\lim_{n\to\infty} s_{n-1} = S$. Since $u_n = s_n - s_{n-1}$, it follows that

$$\lim_{k\to\infty} u_k = \lim_{n\to\infty} u_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = 0. \quad \#$$

4.21.1 COROLLARY. *If* $\lim_{k\to\infty} u_k \neq 0$, *then* $\sum u_k$ *diverges.*

4.21.2 EXAMPLE. Many students seem to have difficulty remembering that if $\lim_{k\to\infty} u_k = 0$, it *does not* follow that the series $\sum u_k$ is convergent. To see this, consider the following series:

$$(1/2 + 1/2) + (1/4 + \cdots + 1/4) + (1/8 + \cdots + 1/8) + \cdots = 1 + 1 + \cdots.$$

[2 terms] [4 terms] [8 terms] etc.

Clearly, the partial sums of the series become large as k increases although we certainly have $\lim_{k\to\infty} u_k = 0$.

4.21.3 EXAMPLE. Show that the series $\sum \frac{k-5}{3k+2}$ diverges.

Solution. Since $\lim_{k\to\infty} \frac{k-5}{3k+2} \neq 0$, we may use Corollary 4.21.1. #

Convergence tests. For an infinite series with nonnegative terms, three simple convergence tests are available, namely, the ***comparison test***, the ***ratio test***, and the ***integral test***. For more general cases, we also have the ***root test*** and the ***alternating series test***.

4.22 THEOREM. *Suppose* $u_k \geq 0$ *for all* k. *If the sequence* (s_n) *of partial sums is bounded, then the series* $\sum u_k$ *converges.*

Proof. Since $u_k \geq 0$, the sequence (s_n) of partial sums is nondecreasing. By Theorem 4.18, (s_n) is convergent, because it is bounded. #

4.22.1 EXAMPLE. Show that the series $\sum 1/k^2$ converges.

Solution. Since $1/k^2 \geq 0$ for all k, Theorem 4.22 applies. We will show that the sequence of partial sums is bounded. We have

$$1 + (1/2^2 + 1/3^2) + (1/4^2 + \cdots + 1/7^2) + (1/8^2 + \cdots + 1/15^2) + \cdots \leq$$

$$\leq 1 + (1/2^2 + 1/2^2) + (1/4^2 + \cdots + 1/4^2) + \cdots =$$

$$= 1 + 1/2 + 1/4 + 1/8 + \cdots = 2$$

Hence, the increasing partial sums is bounded above by 2. #

4.23 THEOREM (COMPARISON TEST). *Let $u_k \geq 0$, $v_k \geq 0$ for all k and $C > 0$. (i) If $u_k \leq C v_k$, and $\sum v_k$ converges, then $\sum u_k$ converges and $\sum u_k \leq C \sum v_k$; (ii) If $C u_k \leq v_k$ and $\sum u_k$ diverges, then $\sum v_k$ diverges.*

Proof. Let s_n and t_n be the nth partial sums of $\sum u_k$ and $\sum v_k$. Then $s_n \leq C t_n$. (*i*) Since (s_n) is nondecreasing and $\lim t_n$ exists, it follows that (s_n) is bounded above by $\lim C t_n$. By Theorem 4.22, the series $\sum u_k$ converges.

(*ii*) If (s_n) were bounded above, it would be convergent; hence, by assumption, (s_n), also $(C s_n)$, is not bounded above. Thus, for any N, we have $t_n \geq C s_n > N$ for large n. Hence, $\lim t_n = \infty$. #

4.23.1 EXAMPLE. Determine convergence or divergence:

(*i*) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ (*ii*) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+7}{2n^4 - n + 3}$

Solutions. (*i*) We shall use the estimation technique used in Example 4.8.2. Since

$$\frac{n^2}{n^3+1} \geq \frac{n^2}{n^3+n^3} = \frac{n^2}{2n^3} = \frac{1}{2}\left(\frac{1}{n}\right),$$

and since $\sum 1/n$ diverges, then so does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$.

(ii) Recall Example 4.8.2 again. Since

$$\frac{n^2+7}{2n^4-n+3} \le \frac{n^2+7n^2}{2n^4-n^4} = \frac{8n^2}{n^4} = 8\left(\frac{1}{n^2}\right).$$

and since $\sum 1/n^2$ converges, then so does $\sum_{n=1}^{\infty} \frac{n^2+7}{2n^4-n+3}$. #

4.24 THEOREM (RATIO TEST). *Let $\sum u_k$ be a series with $u_k \ge 0$ for all k. Assume that there is a number c with $0 < c < 1$ such that $u_{k+1}/u_k \le c$ for large k. Then the series $\sum u_k$ converges*

Proof. Suppose that $u_{k+1}/u_k \le c$, if $k \ge N$. Then

$u_{N+1} \le c\,u_N$, $u_{N+2} \le c\,u_{N+1} \le c^2\,u_N$, etc..

In general, we have $u_{N+n} \le c^n u_N$. Therefore,

$$\sum_{k=N}^{N+n} u_k \le u_N + cu_N + c^2u_N + \cdots + c^n u_N \le$$

$$\le u_N\left(1+c+c^2+c^3+\cdots+c^n\right)$$

$$\le u_N \frac{1}{1-c}.$$

Thus, in effect we have compared our series with the geometric series, (see Example 4.19.2) and we know that the partial sums are bounded. This implies that our series is convergent. #

4.24.1 COROLLARY. *Let $\sum u_k$ be a series with $u_k \ge 0$ for all k. Suppose that* $\lim_{k\to\infty} \frac{u_{k+1}}{u_k} < 1$. *Then the series converges.*

4.24.2 EXAMPLE. *Show that the series $\sum k/3^k$ converges.*

Solution. Let $u_k = k/3^k$. Then as $k \to \infty$, we have

$$\frac{u_{k+1}}{u_k} = \frac{k+1}{3^{k+1}}\frac{3^k}{k} = \frac{k+1}{k}\frac{1}{3} \to \frac{1}{3} < 1.$$

Thus by Corollary 4.24.1, the series is convergent. #

4.25 THEOREM (INTEGRAL TEST). *Let f be a function, which is defined and positive for all $x \geq 1$, and nonincreasing. The series $\sum f(k)$ converges if and only if the improper integral $\int_1^\infty f(x)dx$ converges.*

Proof. Since f is nonincreasing, we have

$f(2) \leq f(x) \leq f(1)$, for $1 \leq x \leq 2$,

$f(3) \leq f(x) \leq f(2)$, for $2 \leq x \leq 3$, etc..

Hence, it follows that

$$f(2) \leq \int_1^2 f(x)dx \leq f(1), \quad f(3) \leq \int_2^3 f(x)dx \leq f(2), \quad \text{etc.}. \qquad (4.25.1)$$

This implies that

$$f(2) + f(3) + \cdots + f(n) \leq \int_1^\infty f(x)dx. \qquad (4.25.2)$$

Hence, if $\int_1^\infty f(x)dx$ converges, then the partial sums of the series $\sum f(k)$ are bounded. Therefore, the series converges, by Theorem 4.22.

To prove the converse, suppose that the series $\sum f(k)$ converges. We use (4.25.1) again to get

$$\int_1^n f(x)dx \leq f(1) + f(2) + \cdots + f(n-1) \leq \sum_{k=1}^\infty f(k). \qquad (4.25.3)$$

Hence, by Theorem 4.18, $\lim_{n\to\infty} \int_1^n f(x)dx$ exists. #

4.25.1 EXAMPLE. Prove that the series $\sum 1/k^2$ converges.

Solution. This is also done in Example 4.22.1. Let $f(x) = 1/x^2$. Then

$$\int_1^\infty f(x)dx = \lim_{n\to\infty} -(1/x)\Big|_1^n = \lim_{n\to\infty}(1 - 1/n) = 1.$$

Thus $\int_1^\infty f(x)dx$ converges. By the Integral Test, $\sum 1/k^2$ is convergent. #

A series $\sum u_k$ (whose terms u_k are possibly negative) is said to ***converge absolutely*** if the series $\sum |u_k|$ converges.

4.26 THEOREM. *A series $\sum u_k$ that converges absolutely is convergent.*

Proof. Define:

$$u_k^+ = \begin{cases} u_k, & \text{if } u_k \geq 0, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad u_k^- = \begin{cases} -u_k, & \text{if } u_k \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then both u_k^+ and u_k^- are nonnegative. Moreover, since

$u_k^+ \leq |u_k|$ and $u_k^- \leq |u_k|$ for all k,

it follows that both $\sum u_k^+$ and $\sum u_k^-$ converge and so does

$$\sum u_k^+ - \sum u_k^- = \sum(u_k^+ - u_k^-) = \sum u_k. \quad \#$$

4.27 THEOREM (ALTERNATING SERIES TEST). *Let $\sum u_k$ be a series such that (i) $|u_k|$ decreases to 0 and (ii) u_k is alternately positive and negative. Then the series is convergent.*

Proof. Let us write the series $\sum u_k$ in the form

$$b_1 - c_1 + b_2 - c_2 + \cdots,$$

with $b_k \geq 0$, and $c_k \geq 0$. Let

$$s_n = b_1 - c_1 + b_2 - c_2 + \cdots + b_n,$$

$$t_n = b_1 - c_1 + b_2 - c_2 + \cdots + b_n - c_n.$$

Since $s_{n+1} = s_n - c_n + b_{n+1}$ and $0 \leq b_{n+1} \leq c_n$, we subtract more from s_n by c_n than we add afterwards by b_{n+1}. Hence, $s_{n+1} \leq s_n$. By a similar argument, we have $t_{n+1} \geq t_n$. Clearly, $s_n \geq t_n$ for all n. Hence, we must have

$$s_1 \geq s_2 \geq s_3 \geq \cdots \geq L \geq \cdots \geq t_3 \geq t_2 \geq t_1.$$

for some real number L. But $\lim (s_n - t_n) = \lim c_n = 0$. Hence, it follows that $\lim s_n = \lim t_n = L$. Therefore, the series converges to L. #

Let $x \in \mathbf{R}$ and let (u_k) be a sequence of real numbers. The series $\sum_{k=1}^{\infty} u_k x^k$ is called a ***power series***.

4.28 THEOREM. *Assume that there is a number $r \geq 0$ such that the series*

$$\sum_{k=1}^{\infty} |u_k| \, r^k$$

converges. Then for all x such that $|x| \leq r$, the power series $\sum_{k=1}^{\infty} u_k x^k$ converges absolutely.

Proof. The absolute value of the term $u_k x^k$ of the given series is $|u_k||x|^k \leq |u_k| r^k$. The conclusion follows from the Comparison Test. #

The ***radius of convergence*** of the power series $\sum_{k=1}^{\infty} u_k x^k$ is the least upper bound of the numbers r for which we have the convergence stated in Theorem 4.28. If there is no such number, then we say that the radius of convergence is infinite.

4.29 THEOREM (ROOT TEST). *Let $\sum u_k x^k$ be a power series and assume that $\lim |u_k|^{1/k} = s$, where $s \in \mathbf{R}$. If $s > 0$ then the radius of convergence of the series is equal to $1/s$. If $s = 0$, the radius of convergence is infinite. If $|u_k|^{1/k}$ becomes arbitrarily large as $k \to \infty$, then the radius of convergence is 0.*

Proof. Without loss, we may assume that $u_k \geq 0$ for all k. Suppose that $s > 0$, and let $0 \leq r < 1/s$. Let $\varepsilon = |rs - 1| > 0$. By hypothesis, the numbers $u_k^{1/k} r$ tend to sr, as $k \to \infty$, and hence (since $rs < 1$) are $< 1 - \varepsilon$ for all sufficiently large k. Hence, the series $\sum u_k r^k$ converges, by comparison with the geometric series $\sum (1-\varepsilon)^k$.

If, on the other hand, $r > 1/s$, then $u_k^{1/k} r$ approaches $sr > 1$, and, hence, we have $u_k^{1/k} r \geq 1 + \varepsilon$ for sufficiently large k. Then comparison with the series $\sum (1-\varepsilon)^k$ shows that the series $\sum u_k r^k$ diverges. We leave the cases $s = 0$ and $s = \infty$ to the reader. #

4.29.1 EXAMPLE. *The series* $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ *has a radius of convergence equal to* 1, *because*

$$\lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{2/n}} = 1. \quad \#$$

REMARK. Experience shows that we can teach part of this material successfully, if we take the trouble to explain and motivate our presentations. On the part of the students, this would require a certain degree of maturity. On our part, it would require a higher level of expertise.

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