# CALCULUS IN PHYSICS

# Remigio Tee

Calculus is probably the most important nathematical tool of physics: it is the primary language of physices. Many of the concepts and laws of physics are most precisely expressed or formulated in the language of calculus. Without calculus Newton would have been unable to formulate his monumental synthesis of the laws of motion (dynamics) and the theory. of universal gravitation. Indeed. Newton invented calculus primarily to settle questions relatcd to his work in physics. Motion involves change and the precise statement of instantancous velocity and instantancous acceleration requires calculus. In his initial analysis of the moon's motion around the earth Newton assumed that the carth and the moon could be treated as point masses attracting cach other through gravity even though both the carth and the moon have nonzero dimensions. After inventing calculus Newton was able to justify this assumption by proving that the gravitational attraction between two spherically symmetric masses is cquivalent to that of two point masses separated by a distance equivalent to the center-to-center distance between the two spherical masses. With caleulus. Newton was then able to derive the orbits of the moon and planets as well as Keplers' three laws of planetary motion. Eclipses, once regarded with fear and terror by the populace for countless millennia had now become predictable physical events. All these because of calculus. Today, calculus and its descendants (differential equations, calculus of variations, integral equations, differential geometry, vector and tensor analysis, etc.) are the everyday working tools of physics. To go into details of even the simplest applications of these fields to physics would require more time and preparation than what we have. We shall therefore limit ourselves to a few common examples.

Newton and the calculus. The first major discovery in the physics of motion was made by Galileo. He discovered the following:

LAW OF INERTIA. No force is required to maintain uniform motion  $(constant speed$  glong a straight path).

th:<br>mi alileo's discovery everyone believed that force is reprion; that a rolling cart, left to itself, would eventuall<br>force is needed to keep it going. Newton, who was is<br>covery everyone believed that force is required to<br>rolling cart, left to itself, would eventually come to<br>reded to keep it going. Newton, who was born the<br>ted. Galileo's law of inertia and restated it as a low el<br>its<br>ri Before Galileo's discovery everyone believed that force is required to otion; that a rolling cart, left to itself, would eventually come force is needed to keep it going. Newton, who was born<br>o died, adopted Galileo's law of inertia and restated it as a ecause force is needed to keep it going. Newton, who was born the<br>Solileo died, adopted, Galileo's law of inertia and restated it as a law because force is needed to keep it going. Newton, who was born the<br>balileo died, adopted Galileo's law of inertia and restated it as a law<br>tion.

R:<br>IO<br>IO. AW OF MOTION. A body at rest will remain at r<br>on will continue moving with constant veloc<br>straight path) unless acted upon by external for N's FIRST LAW OF MOTION. A body at rest will remain at rest<br>in a motion, will continue moving with constant velocity el<br>f and a body in

body in motion will continue moving with constant speed along a straight path) unless acted upon by extensileo's law of inertia states what happens to the body there is no force acting on it. The next logical question is t oee<br>'s<br>is : long a straight path) unless acted upon by external force<br>of inertia states what happens to the body's motion<br>force acting on it. The next logical question is to ask wh ates what happens to the body's mot<br>it. The next logical question is to ask w<br>ing. Newton discovered the law govern ere is no force acting on it. The next logical question is to<br>i when there is force acting. Newton discovered the law<br>tation. This law, now called *Newton's second law of* o force acting on it. The next logical question is to ask where is force acting. Neuton discovered the law governing when there is force acting. Newton discovered the law go<br>ion. This law, now called *Newton's second law of mo*<br>thematically as  $F = ma$ . It says that the effect of an extern ince acting. Newton discovered the law gover<br>now called *Newton's second law of moti* e<br>*uw*<br>is ituation. This law, now called *Newton's second law of motion*, is<br>mathematically as  $F = ma$ . It says that the effect of an external force<br>e motion of a body is to cause it to accelerate (that is, to change its<br>to in such a athematically as  $F = ma$ . It says that the effect of an external forc<br>notion of a body is to cause it to accelerate (that is, to change it<br>i) in such a way that the direction of the acceleration is the same a s F<br>y i<br>tha<br>e a<br>por  $\epsilon = ma$ . It says that the effect of an ext<br>s to cause it to accelerate (that is, to ne motion of a body is to cause it to accelerate (that is, to cha the state of the state of the accelerate (that is, to change its<br>in such a way that the direction of the acceleration is the same as<br>tion of the force and its magnitude is directly proportional to the ha<br>rat<br>v. such a way that the direction of the acceleration is the same as<br>in of the force and its magnitude is directly proportional to the<br>wersely proportional to the mass of the body. th<br>tic<br>t c rection of the force and its magnitude is directly proportional to the and inversely proportional to the mass of the body.<br>t would be noted that the precise mathematical statement of the first the force and its magnitude is d<br>sely proportional to the mass of the<br>e noted that the precise mathemateus of Newton already require call e mass of the bod<br>se mathematical s

d inversely proportional to the mass of the body.<br>
vould be noted that the precise mathematical statement of the first<br>
and laws of Newton already require calculus since the velocities rection of the force and its magnitude is directly proportional to the<br>and inversely proportional to the mass of the body.<br>would be noted that the precise mathematical statement of the fin<br>econd laws of Newton already requ ate<br>sir<br>eo<br>he f 1<br>ve<br>a econd laws of Newton already require calc<br>
inccelerations appearing in the laws are in<br>
ges Thus if the force is not constant the insta no<br>ne<br>ou pp<br>ef<br>ay<br>tic appearing in the laws are instantaneous values, not<br>he force is not constant the instantaneous acceleration is<br>e average acceleration so that calculus is required. For<br>etion the mathematical statements are as follows: al<br>ele<br>pui<br>ov Th<br>ne<br>si ges. Thus if the force is not constant the instantaneous acceleration is<br>
ie same as the average acceleration so that calculus is required. For<br>
imensional motion, the mathematical statements are as follows:<br>
soition..... e<br>de<br>ta cc<br>red<br>ol s the average acceleration so that calculus is r<br>al motion, the mathematical statements are as for ot<br>ata ne mathematical statements are as fol $(t)$ ate<br>. e<br>e

ot<br>position, velocity,  $\frac{1}{t}$  $\epsilon$ acceleration, ،<br>nd<br>nd  $\overline{l}$  $F = ma$ 

ea<br>b<br>b w<br>IW lewton's second law, together with calculus, enables us to determine the force acting on a body when it by by ether with calculus, enables us to det<br>subjected to a given force. Alternatively a body when subjected to a given force. Alternative the force acting on a body when i s to determine the force acting on a body when its motion is eersmine versleid van de versl known.

XAMPLE. Let us determine the motion of an obj et<br>1.<br>3. e<br>oo<br>:1 f an object attached to a<br>a spring or rubber band<br>iven by  $F = -kx$ . or rubber band. According to **Hooke's law**, a spring or runder and the *s* of the retched by an amount x will exert a force given by  $F = -$ <br>plying Newton's second law to this force, we obtain bber band. According to **Hooke's law**, a spring or rubber band by an amount x will exert a force given by  $F = -kx$ .

y :<br>W n<br>|
| iven by .<br>ve obtai<mark>r</mark>

Applying Newton's second law to this force, we obtain  
\n
$$
F = ma,
$$
\n
$$
-kx = m \frac{dv}{dt},
$$
\n
$$
-kx = m \frac{dv}{dx} \frac{dx}{dt}
$$
 (by the chain rule),  
\n
$$
-kx = m \frac{dv}{dx} v,
$$
\n
$$
-kx dx = mvdv,
$$

 $dx = \int_{v_0}^{v} m v dv$  (notice the sloppy notation on the upp

$$
-k\frac{x^2}{2} + k\frac{x_0^2}{2} = \frac{mv^2}{2} - \frac{mv_0^2}{2},
$$

$$
\frac{kx_0^2}{2} + \frac{mv_0^2}{2} = \frac{kx^2}{2} + \frac{mv^2}{2}
$$

st equation above is an example of a *conservation law*. It stat bove is an example of a *conservation law*. It state<br>  $+\frac{1}{2}mv^2$  has the same value for all time. The<br> **over** of the body consisting of the *notential energy* rn<br>ti<br>tio quantity  $\frac{1}{2}kx^2 + \frac{1}{2}mv^2$  has<br>called the *energy* of the body e *energy* of the body consisting of the *potential er*<br>
kinetic energy  $(\frac{1}{2}mv^2)$ . The concepts of work, pote<br>
execution the potential energy V and force F obey  $\cdots$  2  $\cdots$ conce<br>For c the *energy* of the body consisting of the *potential ene*<br>
the *kinetic energy*  $\left(\frac{1}{2}mv^2\right)$ . The concepts of work, poten energy  $\left(\frac{1}{2}mv^2\right)$ . The concept<br>timately interrelated. For cer and force are intimately interrelated. For certain for<br>tive forces) the potential energy  $V$  and force  $R$ <br>hip or<br>are conservative force  $\overline{a}$ and force  $\overline{F}$ 

THE MINDANAO FORUM  
\n
$$
F = -\frac{dV}{dx}
$$
 (one dimension),  $F = -\nabla V$  (two or more dimensions).  
\nThe potential energy V at a point x is defined as the work done by the

int<br>an the observation of the value of  $X$  is defined as the work done by the solution from point x to an arbitrarily fixed reference point x hergy  $V$  at a point  $x$  is defined as the<br>om point  $x$  to an arbitrarily fixed<br>on work done is defined as in going from point  $x$  to nationally, the work done is F in to an arbitrarily fixed point ir<br>Fe ork done is defi

nematically, the work done is defined as  
\n
$$
V = \int_{x}^{x_0} F dx
$$
 (one dimension),  
\n
$$
V = \int_{x}^{x_0} \mathbf{F} \cdot d\ell
$$
 (two or more dimensions).

 $\ddot{\phantom{a}}$ e case of Hooke's law,  $F = -kx$ . The

$$
V = \int_{x}^{x_0} F dx = \int_{x}^{x_0} -kx dx
$$
  
=  $-\frac{1}{2} kx^2 \Big|_{x}^{x_0} = -\frac{1}{2} kx_0^2 + \frac{1}{2} kx^2.$ 

 $\frac{1}{2}$ Efine potential energy for this case, we are otential energy for this case, we arbitrarily choose the  $z_0 = 0$  so that  $V = \frac{1}{2}kx^2$ . The introduction of the conceptions the determination of the motion of the body. Thus, in<br>m  $V = \frac{1}{2}kx^2$ . The introduction of the conditions the determination of the motion of the body. Thus, e i<br>nc ith<br>ly.

energy simplifies the determination of the motion of the body. Thus

\n
$$
E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2.
$$
\n
$$
∴ v = \pm \sqrt{\frac{2E}{m} - \frac{k}{m}x^2} = \pm \sqrt{\frac{2E}{m}}\sqrt{1 - \frac{k}{2E}x^2}.
$$
\n
$$
∴ \frac{dx}{dt} = \pm \sqrt{\frac{2E}{m}}\sqrt{1 - \frac{k}{2E}x^2},
$$
\n
$$
\frac{dx}{\sqrt{1 - \frac{k}{2E}x^2}} = \pm \sqrt{\frac{2E}{m}}dt.
$$

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tegrate, we set  $\sin \theta = \sqrt{\frac{h}{2E}} x$  so  $\frac{1}{2}$ E

$$
x = \sqrt{\frac{2E}{k}} \sin \theta,
$$
  

$$
dx = \sqrt{\frac{2E}{k}} \cos \theta d\theta,
$$

the integral becomes  
\n
$$
\int_{\theta_0}^{\theta} \frac{\sqrt{\frac{2E}{k}} \cos \theta}{\sqrt{1 - \sin^2 \theta}} d\theta = \pm \sqrt{\frac{2E}{m}} \int_0^t dt.
$$
\n
$$
\therefore \quad \int_{\theta_0}^{\theta} d\theta = \pm \sqrt{\frac{k}{m}} t,
$$
\n
$$
\theta - \theta_0 = \pm \sqrt{\frac{k}{m}} t + \theta_0.
$$
\n
$$
\therefore x = \sqrt{\frac{2E}{k}} \sin \theta = \sqrt{\frac{2E}{k}} \sin \left( \pm \sqrt{\frac{k}{m}} t + t_0 \right)
$$

otion of the body is there

In of the body is therefore oscillatory.<br>
SAMPLE. Let us now study the motion of an object moving at a<br>
te in a circular path. (See the figure below.) et<br>ul ai<br>w. te in a circular path. (See the figure belo<br>  $\theta$  in radians

 $\ddot{\phantom{0}}$ 

 $s=r\theta$ ,  $\theta$  in radians,

$$
v = \frac{ds}{dt} = r \frac{d\theta}{dt} = \text{constant},
$$



Figure 1

cation of the mass *m* is given by the vec

 $\int$ <br> $\theta$  $=$  **i**r cos  $\theta$  + **j**r sin  $\theta$ .

e hav<sub></sub>

ferentiating twice, we have  
\n
$$
\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} r \left( -\sin \theta \right) \frac{d\theta}{dt} + \mathbf{j} r (\cos \theta) \frac{d\theta}{dt}, \text{ and}
$$
\n
$$
\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{i} r \left( -\cos \theta \right) \left( \frac{d\theta}{dt} \right)^2 + \mathbf{j} \left( r - \sin \theta \right) \left( \frac{d\theta}{dt} \right)^2
$$
\n
$$
= -\left( \frac{d\theta}{dt} \right)^2 \left[ \mathbf{i} r \cos \theta + \mathbf{j} r \sin \theta \right]
$$
\n
$$
= -\left( \frac{d\theta}{dt} \right)^2 \mathbf{r}
$$
\n
$$
= -\frac{v^2}{r^2} \mathbf{r}.
$$

This says that the direction of the acceleration (and, hence, the apposite that of **r**, that is, toward the center of the circle. It shows the dy moves in a circular orbit at a uniform rate, the force acting rection of the acceleration (and, hence, the force<br>is, toward the center of the circle. It shows that, nd<br>irc<br>f J the that of **r**, that is, toward the center of the circle. It shows may<br>noves in a circular orbit at a uniform rate, the force acting on i<br>directed toward the center of the circular path. In this way<br>discovered that the mo pposite that of **r**, that is, toward the center of the circle. It shows that,<br>bdy moves in a circular orbit at a uniform rate, the force acting on<br>the directed toward the center of the circular path. In this was e directed toward the center of the circular patherment discovered that the moon is being attracted by nets by the sun. scovered that the moon is being attracted by the earth and<br>s by the sun.<br>106 lanets by the sure

REMIGIo TEE orrectly concluding that it is the same force that causes objects to<br>the ground, Newton was able to show that the *force varies as*<br>e square of the distance from the attracting body. He called the<br>**gravity**. All these Newt is the same force that cause bj<br>a Newton was able to show that the *force varies as*<br>the distance from the attracting body. He called the<br>ese Newton did more than 200 years ago. Since than quare of the distance from the attracting body. He called the vity. All these Newton did more then 300 years ago. Since then has rapidly expanded and developed into many areas he distance from the attracting body. He call *ravity*. All these Newton did more then 300 year<br>has rapidly expanded and developed in All these Newton did more then 300 years ago. Since then rapidly expanded and developed into many area<br>sm, relativity, quantum mechanics etc.) where calculus is (electromagnetism, relativity, quantum mechanics, etc.), where calc<br>ectrostatic, for instance, to find the electric fiel<br>c potential  $\phi(\mathbf{r})$  at point **r** due to a continuous has rapidly expanded and developed into magnetism, relativity, quantum mechanics, etc.), where<br>able. In electrostatic, for instance, to find the electric<br>lectrostatic potential  $\phi(\mathbf{r})$  at point r due to a continua n<br>itat<br>le:<br>(r' or<br>(**r**)<br>he ensable. In electrostatic, for instance, to find the electric field  $E(r$  e electrostatic potential  $\phi(r)$  at point r due to a continuous charge  $y \sigma(r')$  one must perform the following volume integrations: electro<br>P<br>Control  $p(r)$  at point **r** differential the following vo ha<br> (**r'**) one must perform the following volume inteq<br>=  $\int_0^{\infty} \frac{\sigma(\mathbf{r}')(\mathbf{r} - \mathbf{r}')dv'}{v}$ 

$$
E(\mathbf{r}) = \int \frac{\sigma(\mathbf{r}')(\mathbf{r} - \mathbf{r}')dv'}{|\mathbf{r} - \mathbf{r}'|^3},
$$

$$
\phi(\mathbf{r}) = \int \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv'.
$$

tegrals are related by the equals  $= -\nabla \phi(\mathbf{r})$ .

$$
\mathbf{E}(\mathbf{r})=-\nabla \phi(\mathbf{r}).
$$

ene<br>One hagnetic field **B** due to a static current dishplicated vector integration involving vect ta<sup>l</sup> <sub>ec</sub><br>s nvolving vec  $\frac{1}{2}$ product.

pr<br>o<br>ir and variational calculus. Minima and r in physics. The laws of the light reflection be unified into one law. va<br><sub>vs</sub> ca<br>vs<br>el na problems often occur in physics. The laws of raction, for instance, can be unified into one law in tysics. The laws of the light refl<br>unified into one law.

or instance, can<br>PRINCIPLE OF L<br>*thar is the one* RIN<br>'nei<br>' PI<br>P<br>N EAST TIME. The path taken by light froi<br>that takes the least amount of travel time ERMAT'S PRINCIPLE OF LEAST TIME. *The path taken by light fit int to another is the one that takes the least amount of travel time*<br>point to another is the one that takes the least amount of travel time

most (if not all) of the laws of physics can be sta<br>
imum or maximum) principles. The level of mat<br>
required is generally higher than those encour if not all) of the laws of physics can be stat<br> $\frac{1}{2}$  can be stat naximum) principles. The level of<br>is generally higher than those en<br>calculus. Whereas extremum proble<br>equire numbers or values for answer equired is generally higher than those encordinary calculus. Whereas extremum problem<br>calculus require numbers or values for answers,<br> $107$ n<br>y<br>r ordinary calculus. Whereas extr<br>
calculus require numbers or valu<br>
107 ot<br>iw  $\overline{c}$ leulus require numbers or values for answers, most ext<br>
107 answers,

 $\frac{S}{\omega}$ roblems in physics require functions for ans<br>the calculus of variations. The initial deve In physics require functions for answers. Such problems of variations. The initial development of this<br>tresult of Bernoulli's solution of the **brachistoch** ers.<br>Pr the calculus of variations. The initial development of this field<br>out as a result of Bernoulli's solution of the **brachistochrom**.<br>problem is this: *Consider two points*  $(x_1,y_1)$  and  $(x_2,y_2)$  in g<br>plane on the earth's s i<sup>t</sup><br>i's of<br> $\frac{d}{dx}$ bout as a result of Bernoulli's solution of the *bra*<br> **n**.<br>
e problem is this: *Consider two points*  $(x_1, y_1)$  and It as a result of Bernoulli's solution of the **brachister**<br>
roblem is this: *Consider two points*  $(x_1,y_1)$  and  $(x_2,$ <br>
ane on the earth's surface. What would be the sha

is this: Consider two points  $(x_1,y_1)$  and  $(x_2,y_2)$  in a<br>the earth's surface. What would be the shape of a<br>primecting the two points, which allows the least travel lane on the earth's surface. What would be the shape of a<br>s slide, connecting the two points, which allows the least travel<br> $(x_1,y_2)$ he earth's surface. What would be the shap<br>nnecting the two points, which allows the leas id<br>T time?



Figure 2

blution is the *cycloid*. An anc the cycloid. An ancient mathematical problem called of similar nature. A typical problem in calculus is of the rectangle enclosing the biggest area for a give ro<br>ca<br>p<br>ve For problem is of similar nature. A typical problem in calculus is to dimensions of the rectangle enclosing the biggest area for a given eter L. The answer is the square with side  $L/4$ . Dido's problem asks f similar nature. A typical problem in calculus is to iti<br>rensions<br>insi nclosing the biggest are<br>with side  $L/4$ . Dido's preatest area for a give ter *L*. The answer is the square with side *L*/4. Dido's problem<br>eometrical figure *will give the greatest area for a given peri*<br>answer is the circle. he answer is the square with side  $L/4$ . Dic<br>cal figure, will give the greatest great for ans<br>Vi he answer is the circ<br>The most common

most common one-dimensional problem in the *calculus of* s to find the curve  $y(x)$  passing through the points  $(x_1,y_1)$  and that the integral or<br>C ore<br>are<br>, t<br>therefore If find the curve  $y(x)$  passing through the points  $(x_1, x_2)$ igure will give the greatest are<br>e circle.<br>nmon one-dimensional proble<br>l the curve  $y(x)$  passing throug  $\frac{1}{10}$ 

$$
I = \int_{(x_1, y_1)}^{(x_2, y_2)} f(y, \dot{y}; x) dx, \text{ where } \dot{y} = \frac{dy}{dx},
$$

|
|
|
|
|
| is extremum. The desired curve  $y(x)$  is given by the **Eul** extremum. The desired curve  $y(x)$  is given by the **Eul**  $\int f$ equation

tion  
\n
$$
\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) - \frac{\partial f}{\partial y} = 0.
$$

ultidimensional version of the Eule<br>1990 er:<br>i qu<br>-

$$
\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_k}\right) - \frac{\partial f}{\partial y_k} = 0, \qquad k = 1, 2, 3, ..., n.
$$

 $\left(\frac{\partial f}{\partial y_k}\right) - \frac{\partial f}{\partial y_k} = 0,$   $k = 1, 2, 3, ..., n$ .<br>EXAMPLE. Let us find the curve with the shortest lengtwo points  $p_1$  and  $p_2$  in a plane. Thus, et.<br>P

ecting two points 
$$
p_1
$$
 and  $p_2$  in a plane. Thus,  
\n
$$
I = \int_{p_1}^{p_2} ds = \int \sqrt{(dx)^2 + (dy)^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int \sqrt{1 + y^2} dx
$$
, and  
\n
$$
f = \sqrt{1 + y^2} = \left(1 + y^2\right)^{\frac{1}{2}}.
$$

Euler-Lagrange equation give:  
\n
$$
\frac{d}{dx} \left( \frac{\partial f}{\partial \dot{y}} \right) - \frac{\partial f}{\partial y} = 0,
$$
\n
$$
\frac{d}{dx} \left( \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} \right) - 0 = 0, \text{ or}
$$
\n
$$
\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = \text{constant} = \frac{1}{a},
$$

so that

$$
a2 y2 = 1 + y2,
$$
  
\n
$$
y2 = \frac{1}{a2 - 1} = b2,
$$
  
\n
$$
y = b,
$$
  
\n
$$
\frac{dy}{dx} = b,
$$
  
\n
$$
y = bx + c.
$$

.<br>In e shortest path connecting two points in a plane is a strategies of the state vo<br>d by Instants *b* and *c* are determined by making the  $\frac{P^{\text{target}}}{\text{line}}$  is a strategient nd c are determined by making the line is a strain<br>discovered a reformulation of  $\chi$ .

wo points.<br>
n 1834 Hamilton discovered a reformula<br>
variational principle. This is a more gen ew<br>m<br>**un** milton discovered a reformulation of Newton's second  $l_{\text{av}}$ <br>principle. This is a more general reformulation  $\sin_{\text{c}}\theta_{\text{av}}$ <br>extended to other areas of physics outside  $N_{\text{ewton}}$ .<br>amilton's formulation, a *Lagrangian* extended to oth<br>
panics. In Hamilton's formul<br>
efined to be the difference reas of physics  $_{\text{out}_s}$ <br>a **Lagrangian** f ec<sub>ol</sub><br>si<sub>r</sub><br>Vewton between two points on the path of the effined by  $\mathbf{L}$ particle is defined by Hamilton's formulation, a *Lagrangian function*  $L \approx T$ <br>be the difference between the Kinetic energy  $T$  and<br>gy V. The *action* S between two points on the path of<br>ned by nciple. This i<br>xtended to o<br>ilton's formu nd<br><sup>nce</sup><br>F7 ifference between the Kinetic energy  $T = T$ be the difference between the Kinetic energy  $T$  at gy V. The *action* S between two points on the path ned by

$$
S = \int_{t_1}^{t_2} L \, dt = \int_{t_1}^{t_2} L(x, \dot{x}; t) \, dt
$$

and  $t_2$  determine the first and second points, respectively<br>a ron's PRINCIPLE *Out of the infinitely-many* 

secor<br>the *i*<br>ual p<br>r. for i Efirst and second points, respectively.<br> *C. Out of the infinitely-many possible curt 2, the actual path*  $x(t)$  *taken by the part i.* RII<br>Ini<br>ar Put<br>th<br>n  $\int$  the  $\int$ infi oss<br>the<br>sta irves<br><sup>cle</sup> is cting point 1 and point 2, the actual path  $x(t)$  taken by the particle is<br>rve that gives an extremum value for the action S.<br> $\frac{1}{2}$ ves an extremum value for the acti

lu<br>**ii** This was originally called *Hamilton's principle of least action* since<br>ost cases *S* is minimum. In this form the Euler-Lagrange equation<br>mes as originally called *Hamilton's principle of least action*<br>es S is minimum. In this form the Fuler-Lagrange eq ino<br>he equa<br>-<br>becomes  $\bar{r}$ 

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.
$$

ust be emphasized that the coordinate  $x$  may not nec nat the coordinate  $x$  mational form with generalize equation becomes necessari<br>Districts multi-dimensional form with generalized coordinates  $q_1$ ,  $q_2$ .<br>Euler-Lagrange equation becomes rm<br>tio ,  $q_n$  the Euler-Lagrange equation beco<br>  $d \left( \partial L \right) = \partial L$ qua<br>-

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0.
$$

w can be derived from the ever the control of  $\mathfrak{g}$ Lagrangian

$$
L = \frac{1}{2}mv^2 - V(x) = \frac{1}{2}m\dot{x}^2 - V(x).
$$

Thus.

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0,
$$
\n
$$
\frac{d}{dt}(m\dot{x}) + \frac{\partial V}{\partial x} = 0, \text{ or}
$$
\n
$$
\frac{-\partial V}{\partial x} = \frac{d}{dt}(m\dot{x}) = m\ddot{x}.
$$

But  $\ddot{x}$  is the acceleration and  $\frac{-\partial V}{\partial x} = F$ , the force, so that  $F = ma$ .

Hamilton's principle serves as the unifying principle of the entire physical universe. In electomagnetism, the path of the particle is obtained from the Lagrangian

$$
L = \frac{1}{2}mv^2 - q\phi + q\mathbf{v} \cdot \mathbf{A} ,
$$

where  $\phi$  is the scalar potential and A is the magnetic vector potential. Quantum field theory is formulated entirely in the Lagrangian Hamiltonian formalisms and the general theory of relativity can also be incorporated in this approach.

**Einstein and relativity.** In the hands of Einstein, Newton's theory of gravitation became a geometrical theory called the theory of general relativity. The formulation of this theory requires Riemannian geometry and the special theory of relativity.

Riemannian geometry requires that the geometry of space near any given point is Euclidean, meaning the Pythagorean theorem holds in the neighborhood of that point. In two-dimensional Riemannian geometry, this implies that one can construct in the neighborhood of any point a local Cartesian coordinate system  $x$ ,  $y$  such that

 $(ds)^2 = (dx)^2 + (dy)^2$ .

In polar coordinates  $(r, \theta)$ , this metric is given by

$$
(ds)^2 = (dr)^2 + r^2 (d\theta)^2.
$$



# Figure 3

Both Cartesian and polar coordinate systems are said to be *orthogonal* systems where cross-terms such as  $dxdy$  or  $drd\theta$  do not occur in the metric or distance formula  $(ds)^2$ . To illustrate a non-orthogonal system, consid



Figure 4

By the cosine law,

$$
(ds)^{2} = (du)^{2} + (dv)^{2} - 2\cos\theta \, du dv.
$$

In this metric a cross-term occurs which disappears when  $\theta = 90^{\circ}$ , i.e., when  $u$  and  $v$  are orthogonal.

is In *n*-dimensional Riemannian geometry the general form of the metric

$$
(ds)^2 = \sum_i \sum_j g_{ij} du_i du_j.
$$

re educes to

$$
(ds)^2 = \sum_i g_{ii} du_i du_i.
$$

yster<br>ure<br>e is<br>gensio<br>, ma nature is, ho<br>
The geometric<br>
impropensional<br>
impropensional hower.<br>
in<br>
in Ei<br>ps<br>ai<br>ai ne<br>f<br>e. governed by<br>metric is detern<br>bonal space the pse<br>itic<br>ing<br>alc<br>eoc meth<br>
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s sp<br>
f eq<br>
f eq<br>
dz<br>
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ds<br>
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ds eo<br>is<br>d an geometry whose metric is determined by the distribution of<br>
in this four-dimensional space there is no such thing as<br>
mal force. Instead, material bodies simply behave according to<br>
plized law of inertia where the conce expometry whose metric is determined by the d<br>his four-dimensional space there is no su<br>force Instead material bodies simply behave etermined by the disy<br>no<br>ns f<br>s<br>d<br>a four<br>he<br>hinding orc<br>la<br>tt Instead, material bodies simply behave according to<br>
finertia where the concept of constant speed along a<br>
red by a four-dimensional speed along a geodesic<br>
nt vector). In this space nothing is at rest since the nat<br>wh<br>1 f neralized law of inertia where the concept of coraw of inertia where the concept of constant speed alone<br>placed by a four-dimensional speed along a geod<br>angent vector). In this space nothing is at rest since<br>llways moves. oe<br>d<br>cs along ne is replaced by a four-dimensional speed along a g<br>nstant tangent vector). In this space nothing is at rest si<br>dinate always moves. pe<br>in<br>e,<br>a c constant tangent vector). In this space nothing is at rest sin<br>ordinate always moves. (constant tangent vector). In this space nothing is a<br>
sordinate always moves.<br>
cording to Einstein's principle of equivalence, at an<br>
ir-dimensional spacetime one can construct a coord

Einstein's principle of equivalence, at any given point in onal spacetime one can construct a coordinate system  $x$ , in lw<br>E<br>on c<br>c in<br>e uivalence, at any t a<br>or the four-<br> $\left(\frac{1}{t}\right)^2$ c<sub>o</sub> ys  $\mu$ ,  $t$  such that

$$
(ds)^{2} = c^{2}(dt)^{2} - (dx)^{2} - (dy)^{2} - (dz)^{2},
$$

is the speed of light and t is time. This is called a **local ine**<br>
system. Unlike the Riemannian metric, the Einstein many<br>
whe positive negative or zero. These correspond to resting the speed of light and  $t$  is time. This is called a *lot* system. Unlike the Riemannian metric the Einst ys<br>po<br>di<br>di Unlike the Riemannian metric, the Einstein me, negative or zero. These correspond to particles veater than  $c$ , or equal to  $c$ , respectively. The sec ne<br>co:<br>?, may be positive, negative or zero. These correspond to par ay be positive, negative or zero. These correspond to particles with<br>ess than c, greater than c, or equal to c, respectively. The secon-<br>ity is disallowed by special relativity.<br>e Einsteinian metric has the general form is disallowed by special relativity.<br>insteinian metric has the general form<br>=  $\sum \sum g_{ij} du_i du_j$ . possibility is disallowed by special relativity.

ibility is disallowed by sp  
The Einsteinian metric has  

$$
(ds)^2 = \sum_i \sum_j g_{ij} du_i du_j.
$$

 $\begin{array}{c} \n\text{veer} \\
\hline\n\end{array}$  $\int (s)^2 \ge 0$ , the distance between two points along a given curve  $u_k(s)$  is by

$$
s = \int ds = \int \sqrt{\sum_{i} \sum_{j} g_{ij} du_i du_j}
$$

$$
= \int \sqrt{\sum_{i} \sum_{j} g_{ij} \frac{du_i}{ds} \frac{du_j}{ds}} ds
$$

THE MINDANAO FORUM<br>
the geodesic (or shortest curve) between the two point<br>
the extremum. The quation then gives the geodesic equal<br>  $\sum \sum \Gamma_{i}^{k} \frac{du_{i}}{dt} \frac{du_{j}}{dt} = 0$ . hortest curve) between the two points, one<br>to s to obtain the extremum. The  $Euler$ -<br>he geodesic equation alculus of variations to *s* to obtain the extremum. The  $E_{ul}$ <br>equation then gives the geodesic equation  $\mathbf{h}$ 

$$
\frac{d^2u_k}{ds^2} + \sum_i \sum_j \Gamma_{ij}^k \frac{du_i}{ds} \frac{du_j}{ds} = 0.
$$

at spaces there exist global inertial coordinate systems. In such a nate system  $\Gamma_{ij}^k = 0$  so that the geodesic equation becomes stem  $\Gamma_{ij}^k = 0$  so that the geodesic equation beco<br>).

$$
\frac{d^2u_k}{ds^2} = 0.
$$

just the equation of a straight line. Thus for flat spaces the geod<br>uces to Galileo's law of inertia.<br>a geometric theory of gravity proposed by Finstein is known a is just the equation of a straigh<br>duces to Galileo's law of iner<br>he geometric theory of gravit

s to Galileo's law of inertia.<br>
ometric theory of gravity proposed by Einstein is known as the<br> *Fory of relativity*. Once the metric of the space is found the eory of gravity proposed by Einstein is known as the *lativity*. Once the metric of the space is found the determined. In this space, material bodies simply of relativity. Once the metric of the space is<br>asily determined. In this space, material bodi<br>sic curves. The planet, for instance, follow geod are easily determined. In this space, materies geodesic curves. The planet, for instance, follo mensional space time as they orbit around the sum is the metric determined in a given distribution external vector and the space, material bodies simply<br>odesic curves. The planet, for instance, follow geodesic paths<br>isional space time as they orbit around the sun.<br>the metric determined in a given distribution of matter n<br>et<br>in ur-dimensional space time as they orbit arou<br>How is the metric determined in a given

the metric determined in a given distribution of matter or<br>metric is governed by the **Finstein field equation** etermined in a given dist<br>erned by the *Einstein field*  $\frac{c}{E}$  $\int$ the metric is governed by the *Einstein field equal*<br>  $\frac{1}{2}$   $\frac{1}{$  $\text{inc}\atop \text{go}\ \pi( \frac{\pi}{3}$ me as they orbit around the sun.<br>termined in a given distribution of

$$
R_{ij} - \frac{1}{2} g_{ij} R = + \frac{8 \pi G}{c^3} T_{ij},
$$

 $y =$  Ricci tensor,  $R =$  scalar curvature,  $T_{ij} =$  energies Lagrangian-Hamiltonian formulation allows tensor.

tensor. The Lagrangian-Hamiltonian formulation<br>generalized coordinates  $q_1, q_2, q_3, ..., q_n$ . all<br>T of for:<br> $q_3$ , allows the use Their corresponding coordinates  $q_1$ ,  $q_2$ , omenta are defined by  $q_3, \ldots, q_n$ 

$$
p_k = \frac{\partial L}{\partial \dot{q}_k},
$$

the Lagrangian of the system. For classical physics (med<br>
etism),  $q_k$  and  $p_k$  are numbers so that<br>  $\frac{114}{114}$  $\boldsymbol{k}$ nd  $p_k$  are nun

the (mechanics,

$$
a_k p_k - p_k q_k = 0.
$$

ene<br>y t nd the general<br>independent of the set of the s not necessarily numbers and<br>  $n_k = i\hbar$ ,<br>
and  $\hbar = \text{Planck's operator}$ 

$$
q_k p_k - p_k q_k = i\hbar,
$$

Follanck's constant divided by<br>
ations of  $q_k$  and  $p_k$ , while Schro<br>
ation of these quantities. In the Planck's constant divided by  $2\pi$ . atrix repi<br>us repre fed<br>and<br>and  $k$  and  $p_k$ , while Sch<br>hese quantities. In bt<br>ed er<br>9 the<br>d uantities. In the Sc  $\sum_{k=1}^{\infty}$  is a number and

$$
p_k = \frac{\hbar}{i} \frac{\partial}{\partial q_k}.
$$

lergy *E* is represented by  $i\hbar \frac{\partial}{\partial \tau}$ . These quantities are suppose or act on the wave function  $\Psi$ . Thus, for the harmonic gy *E* is represented by  $i\hbar \frac{\partial}{\partial \tau}$ . These quantities are supposed to r act on the wave function  $\Psi$ . Thus, for the harmonic oscillator ergy is given by  $\frac{1}{2}$ . sc ergy is given by<br> $\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{2}{2}$ 

$$
E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2
$$
  
=  $\frac{(mv)^2}{2m} + \frac{1}{2}kx^2$   
=  $\frac{p^2}{2m} + \frac{1}{2}kx^2$ ,  $p = mv$  = momentum,

pr antum-mechanical version in the Sch<br>  $\partial \Psi = \hbar^2 d^2 \Psi$ epi<br>C  $\frac{1}{2}$ 

$$
\text{equation:}\n\begin{aligned}\n\text{Equation:}\n\frac{\partial \psi}{\partial t} &= -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2} kx^2 \psi.\n\end{aligned}
$$
\n
$$
\text{This is called the Schroedingel.\n
$$
\nEquation: Many problems in atomic

=  $-\frac{n}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2}kx^2 \psi$ .<br>called the *Schroedinger equation for the* the *Schroedinger equation for the quantum harmonic*<br>roblems in atomic, molecular and solid state physics are<br>ume prescription. <mark>qu</mark><br>mo ny problems in a<br>ne same prescript at<br>  $he$ 

One cannot possibly hope to get a deep understanding of physics without mastering calculus. The preceding survey gives only a glimpse of a very limited area, yet, it already reveals the great power of calculus.