CALCULUS IN PHYSICS

Remigio Tee

Calculus is probably the most important mathematical tool of physics; it is the primary language of physics. Many of the concepts and laws of physics are most precisely expressed or formulated in the language of calculus. Without calculus Newton would have been unable to formulate his monumental synthesis of the laws of motion (dynamics) and the theory. of universal gravitation. Indeed, Newton invented calculus primarily to settle questions related to his work in physics. Motion involves change and the precise statement of instantaneous velocity and instantaneous acceleration requires calculus. In his initial analysis of the moon's motion around the earth Newton assumed that the earth and the moon could be treated as point masses attracting each other through gravity even though both the earth and the moon have nonzero dimensions. After inventing calculus Newton was able to justify this assumption by proving that the gravitational attraction between two spherically symmetric masses is equivalent to that of two point masses separated by a distance equivalent to the center-to-center distance between the two spherical masses. With calculus, Newton was then able to derive the orbits of the moon and planets as well as Keplers' three laws of planetary motion. Eclipses, once regarded with fear and terror by the populace for countless millennia had now become predictable physical events. All these because of calculus. Today, calculus and its descendants (differential equations, calculus of variations, integral equations, differential geometry, vector and tensor analysis, etc.) are the everyday working tools of physics. To go into details of even the simplest applications of these fields to physics would require more time and preparation than what we have. We shall therefore limit ourselves to a few common examples.

Newton and the calculus. The first major discovery in the physics of motion was made by Galileo. He discovered the following:

LAW OF INERTIA. *No force is required to maintain uniform motion (constant speed along a straight path).*

Before Galileo's discovery everyone believed that force is required to maintain motion; that a rolling cart, left to itself, would eventually come to rest because force is needed to keep it going. Newton, who was born the year Galileo died, adopted Galileo's law of inertia and restated it as a law of motion.

NEWTON'S FIRST LAW OF MOTION. *A body at rest will remain at rest and a body in motion will continue moving with constant velocity (constant speed along a straight path) unless acted upon by external force.*

Galileo's law of inertia states what happens to the body's motion when there is no force acting on it. The next logical question is to ask what happens when there is force acting. Newton discovered the law governing this situation. This law, now called ***Newton's second law of motion***, is stated mathematically as $F = ma$. It says that the effect of an external force on the motion of a body is to cause it to accelerate (that is, to change its velocity) in such a way that the direction of the acceleration is the same as the direction of the force and its magnitude is directly proportional to the force and inversely proportional to the mass of the body.

It would be noted that the precise mathematical statement of the first and second laws of Newton already require calculus since the velocities and accelerations appearing in the laws are instantaneous values, not averages. Thus if the force is not constant the instantaneous acceleration is not the same as the average acceleration so that calculus is required. For one-dimensional motion, the mathematical statements are as follows:

position, $x = x(t)$

velocity, $v = \frac{dx}{dt}$

acceleration, $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Newton's second law, $F = ma$

Newton's second law, together with calculus, enables us to determine the motion of a body when subjected to a given force. Alternatively, it allows us to determine the force acting on a body when its motion is known.

5.1 EXAMPLE. Let us determine the motion of an object attached to a spring or rubber band. According to ***Hooke's law***, a spring or rubber band when stretched by an amount x will exert a force given by $F = -kx$.

Applying Newton's second law to this force, we obtain

$$F = ma,$$

$$-kx = m\frac{dv}{dt},$$

$$-kx = m\frac{dv}{dx}\frac{dx}{dt} \quad \text{(by the chain rule)},$$

$$-kx = m\frac{dv}{dx}v,$$

$$-kxdx = mvdv,$$

$$\int_{x_0}^{x} -kxdx = \int_{v_0}^{v} mvdv \quad \text{(notice the sloppy notation on the upper limits)},$$

$$-k\frac{x^2}{2} + k\frac{x_0^2}{2} = \frac{mv^2}{2} - \frac{mv_0^2}{2},$$

$$\frac{kx_0^2}{2} + \frac{mv_0^2}{2} = \frac{kx^2}{2} + \frac{mv^2}{2}.$$

The last equation above is an example of a ***conservation law***. It states that the quantity $\frac{1}{2}kx^2 + \frac{1}{2}mv^2$ has the same value for all time. This value is called the ***energy*** of the body consisting of the ***potential energy*** $\left(\frac{1}{2}kx^2\right)$ and the ***kinetic energy*** $\left(\frac{1}{2}mv^2\right)$. The concepts of work, potential energy and force are intimately interrelated. For certain forces (called conservative forces) the potential energy V and force F obey the relationship

$$F = -\frac{dV}{dx} \quad \text{(one dimension)}, \qquad \boldsymbol{F} = -\nabla \boldsymbol{V} \text{ (two or more dimensions)}.$$

The potential energy V at a point x is defined as the work done by the force F in going from point x to an arbitrarily fixed reference point x_0. Mathematically, the work done is defined as

$$V = \int_x^{x_0} F dx \quad \text{(one dimension)},$$

$$\boldsymbol{V} = \int_x^{x_0} \boldsymbol{F} \cdot d\boldsymbol{\ell} \quad \text{(two or more dimensions)}.$$

In the case of Hooke's law, $F = -kx$. Therefore,

$$V = \int_x^{x_0} F dx = \int_x^{x_0} -kx dx$$

$$= -\frac{1}{2}kx^2 \Big|_x^{x_0} = -\frac{1}{2}kx_0^2 + \frac{1}{2}kx^2 .$$

To define potential energy for this case, we arbitrarily choose the reference point $x_0 = 0$ so that $V = \frac{1}{2}kx^2$. The introduction of the concept of energy simplifies the determination of the motion of the body. Thus,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 .$$

$$\therefore \quad v = \pm\sqrt{\frac{2E}{m} - \frac{k}{m}x^2} = \pm\sqrt{\frac{2E}{m}}\sqrt{1 - \frac{k}{2E}x^2} .$$

$$\therefore \quad \frac{dx}{dt} = \pm\sqrt{\frac{2E}{m}}\sqrt{1 - \frac{k}{2E}x^2} ,$$

$$\frac{dx}{\sqrt{1 - \frac{k}{2E}x^2}} = \pm\sqrt{\frac{2E}{m}}dt .$$

To integrate, we set $\sin\theta = \sqrt{\frac{k}{2E}}\,x$ so that

$$x = \sqrt{\frac{2E}{k}}\sin\theta,$$

$$dx = \sqrt{\frac{2E}{k}}\cos\theta\, d\theta,$$

and the integral becomes

$$\int_{\theta_0}^{\theta} \frac{\sqrt{\frac{2E}{k}}\cos\theta}{\sqrt{1-\sin^2\theta}}\, d\theta = \pm\sqrt{\frac{2E}{m}}\int_0^t dt.$$

$$\therefore \int_{\theta_0}^{\theta} d\theta = \pm\sqrt{\frac{k}{m}}\,t,$$

$$\theta - \theta_0 = \pm\sqrt{\frac{k}{m}}\,t,$$

$$\theta = \pm\sqrt{\frac{k}{m}}\,t + \theta_0.$$

$$\therefore x = \sqrt{\frac{2E}{k}}\sin\theta = \sqrt{\frac{2E}{k}}\sin\left(\pm\sqrt{\frac{k}{m}}\,t + t_0\right).$$

The motion of the body is therefore oscillatory.

5.2 EXAMPLE. Let us now study the motion of an object moving at a constant rate in a circular path. (See the figure below.)

$s = r\theta$, θ in radians,

$$v = \frac{ds}{dt} = r\,\frac{d\theta}{dt} = \text{ constant},$$

$$\frac{d\theta}{dt} = \frac{v}{r} = \text{constant.}$$

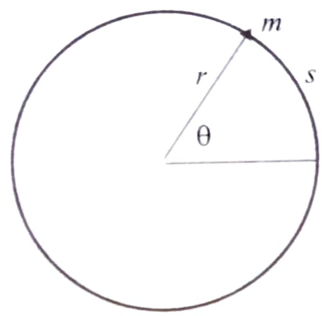

Figure 1

The location of the mass m is given by the vector

$$\mathbf{r} = \mathbf{i}r\cos\theta + \mathbf{j}r\sin\theta\,.$$

Differentiating twice, we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}\,r\,(-\sin\theta)\frac{d\theta}{dt} + \mathbf{j}r(\cos\theta)\frac{d\theta}{dt}, \text{ and}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{i}r\,(-\cos\theta)\left(\frac{d\theta}{dt}\right)^2 + \mathbf{j}\,r(-\sin\theta)\left(\frac{d\theta}{dt}\right)^2$$

$$= -\left(\frac{d\theta}{dt}\right)^2\left[\mathbf{i}\,r\cos\theta + \mathbf{j}\,r\sin\theta\right]$$

$$= -\left(\frac{d\theta}{dt}\right)^2\mathbf{r}$$

$$= -\frac{v^2}{r^2}\,\mathbf{r}\,.$$

This says that the direction of the acceleration (and, hence, the force) is opposite that of **r**, that is, toward the center of the circle. It shows that, if a body moves in a circular orbit at a uniform rate, the force acting on it must be directed toward the center of the circular path. In this way, Newton discovered that the moon is being attracted by the earth and the planets by the sun.

Correctly concluding that it is the same force that causes objects to fall to the ground, Newton was able to show that the *force varies as inverse square of the distance from the attracting body*. He called the force ***gravity***. All these Newton did more then 300 years ago. Since then physics has rapidly expanded and developed into many areas (electromagnetism, relativity, quantum mechanics, etc.), where calculus is indispensable. In electrostatic, for instance, to find the electric field $\mathbf{E}(\mathbf{r})$ and the electrostatic potential $\phi(\mathbf{r})$ at point $\mathbf{r}$ due to a continuous charge density $\sigma(\mathbf{r}')$ one must perform the following volume integrations:

$$\mathbf{E}(\mathbf{r}) = \int \frac{\sigma(\mathbf{r}')(\mathbf{r} - \mathbf{r}')dv'}{|\mathbf{r} - \mathbf{r}'|^3},$$

$$\phi(\mathbf{r}) = \int \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv'.$$

These integrals are related by the equation

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}).$$

To obtain the magnetic field $\mathbf{B}$ due to a static current distribution requires a more complicated vector integration involving vector cross-product.

Extremum principles and variational calculus. Minima and maxima problems often occur in physics. The laws of the light reflection and refraction, for instance, can be unified into one law.

FERMAT'S PRINCIPLE OF LEAST TIME. *The path taken by light from one point to another is the one that takes the least amount of travel time.*

Indeed, most (if not all) of the laws of physics can be stated as extremum (minimum or maximum) principles. The level of mathematical sophistication required is generally higher than those encountered in introductory or ordinary calculus. Whereas extremum problems ordinarily encountered in calculus require numbers or values for answers, most extremum problems in physics require functions for answers. Such problems require the calculus of variations. The initial development of this field came about as a result of Bernoulli's solution of the ***brachistochrome problem***.

The problem is this: *Consider two points* (x_1,y_1) *and* (x_2,y_2) *in a vertical plane on the earth's surface. What would be the shape of a frictionless slide, connecting the two points, which allows the least travel time?*

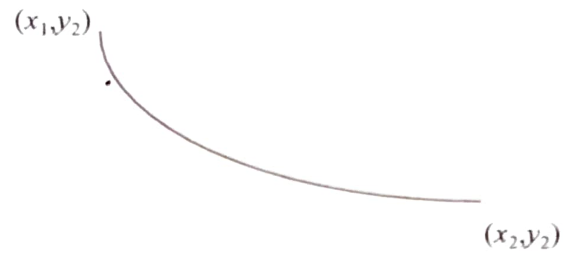

Figure 2

The solution is the *cycloid*. An ancient mathematical problem called ***Dido's problem*** is of similar nature. A typical problem in calculus is to find the dimensions of the rectangle enclosing the biggest area for a given perimeter L. The answer is the square with side $L/4$. Dido's problem asks *what geometrical figure will give the greatest area for a given perimeter* L. The answer is the circle.

The most common one-dimensional problem in the *calculus of variation* is to find the curve $y(x)$ passing through the points (x_1,y_1) and (x_2,y_2) such that the integral

$$I = \int_{(x_1,y_1)}^{(x_2,y_2)} f(y,\dot{y};x)\,dx, \quad \text{where } \dot{y} = \frac{dy}{dx},$$

is an extremum. The desired curve $y(x)$ is given by the ***Euler-Lagrange equation***

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) - \frac{\partial f}{\partial y} = 0.$$

The multidimensional version of the Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_k}\right) - \frac{\partial f}{\partial y_k} = 0, \qquad k = 1,\ 2,\ 3,\ \ldots,\ n.$$

5.3 EXAMPLE. Let us find the curve with the shortest length connecting two points p_1 and p_2 in a plane. Thus,

$$I = \int_{p_1}^{p_2} ds = \int \sqrt{(dx)^2 + (dy)^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{1 + \dot{y}^2}\, dx, \text{ and}$$

$$f = \sqrt{1 + \dot{y}^2} = \left(1 + \dot{y}^2\right)^{1/2}.$$

The Euler-Lagrange equation gives

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) - \frac{\partial f}{\partial y} = 0,$$

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}\right) - 0 = 0, \quad \text{or}$$

$$\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = \text{constant} \equiv \frac{1}{a},$$

so that

$$a^2 \dot{y}^2 = 1 + \dot{y}^2,$$

$$\dot{y}^2 = \frac{1}{a^2 - 1} \equiv b^2,$$

$$\dot{y} = b,$$

$$\frac{dy}{dx} = b,$$

$$y = bx + c.$$

Thus the shortest path connecting two points in a plane is a straight line. The constants b and c are determined by making the line pass through the two points.

In 1834 Hamilton discovered a reformulation of Newton's second law as a variational principle. This is a more general reformulation since the principle can be extended to other areas of physics outside Newtonian mechanics. In Hamilton's formulation, a ***Lagrangian function*** $L = T - V$ is defined to be the difference between the Kinetic energy T and the potential energy V. The ***action*** S between two points on the path of the particle is defined by

$$S = \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} L(x, \dot{x}; t)\, dt ,$$

where t_1 and t_2 determine the first and second points, respectively.

HAMILTON'S PRINCIPLE. *Out of the infinitely-many possible curves connecting point* 1 *and point* 2, *the actual path* $x(t)$ *taken by the particle is the curve that gives an extremum value for the action* S.

This was originally called ***Hamilton's principle of least action*** since in most cases S is minimum. In this form the Euler-Lagrange equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 .$$

It must be emphasized that the coordinate x may not necessarily be Cartesian. In multi-dimensional form with generalized coordinates q_1, q_2, q_3, …, q_n the Euler-Lagrange equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 .$$

The one-dimensional Newton's second law can be derived from the Lagrangian

$$L = \frac{1}{2}mv^2 - V(x) = \frac{1}{2}m\dot{x}^2 - V(x).$$

Thus,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0,$$

$$\frac{d}{dt}(m\dot{x}) + \frac{\partial V}{\partial x} = 0, \quad \text{or}$$

$$\frac{-\partial V}{\partial x} = \frac{d}{dt}(m\dot{x}) = m\ddot{x}.$$

But $\ddot{x}$ is the acceleration and $\frac{-\partial V}{\partial x} = F$, the force, so that $F = ma$.

Hamilton's principle serves as the unifying principle of the entire physical universe. In electomagnetism, the path of the particle is obtained from the Lagrangian

$$L = \frac{1}{2}mv^2 - q\phi + q\mathbf{v} \cdot \mathbf{A},$$

where ϕ is the scalar potential and **A** is the magnetic vector potential. Quantum field theory is formulated entirely in the Lagrangian-Hamiltonian formalisms and the general theory of relativity can also be incorporated in this approach.

Einstein and relativity. In the hands of Einstein, Newton's theory of gravitation became a geometrical theory called the *theory of general relativity*. The formulation of this theory requires Riemannian geometry and the special theory of relativity.

Riemannian geometry requires that the geometry of space near any given point is Euclidean, meaning the *Pythagorean theorem* holds in the neighborhood of that point. In two-dimensional Riemannian geometry, this implies that one can construct in the neighborhood of any point a local Cartesian coordinate system x, y such that

$$(ds)^2 = (dx)^2 + (dy)^2.$$

In polar coordinates (r, θ), this metric is given by

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2.$$

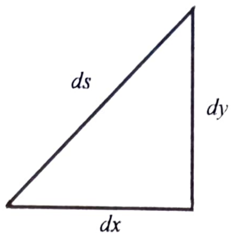

Figure 3

Both Cartesian and polar coordinate systems are said to be *orthogonal systems* where cross-terms such as $dxdy$ or $drd\theta$ do not occur in the metric or distance formula $(ds)^2$. To illustrate a non-orthogonal system, consider the oblique coordinates u, v as shown below.

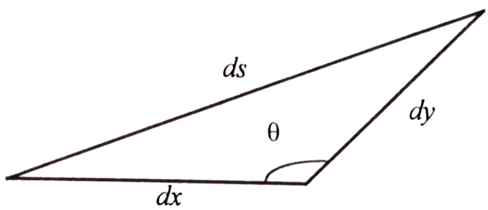

Figure 4

By the cosine law,

$$(ds)^2 = (du)^2 + (dv)^2 - 2\cos\theta\, dudv.$$

In this metric a cross-term occurs which disappears when $\theta = 90^0$, i.e., when u and v are orthogonal.

In n-dimensional Riemannian geometry the general form of the metric is

$$(ds)^2 = \sum_i \sum_j g_{ij}\, du_i du_j.$$

For orthogonal coordinate systems the metric reduces to

$$(ds)^2 = \sum_i g_{ii} du_i du_i \,.$$

The geometry of nature is, however, not Riemannian. Einstein discovered that the universe is governed by a four-dimensional pseudo-Riemannian geometry whose metric is determined by the distribution of matter. In this four-dimensional space there is no such thing as gravitational force. Instead, material bodies simply behave according to the generalized law of inertia where the concept of constant speed along a straight line is replaced by a four-dimensional speed along a geodesic curve (constant tangent vector). In this space nothing is at rest since the time coordinate always moves.

According to Einstein's principle of equivalence, at any given point in the four-dimensional spacetime one can construct a coordinate system x, y, z, t such that

$$(ds)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \,,$$

where c is the speed of light and t is time. This is called a ***local inertial coordinate system***. Unlike the Riemannian metric, the Einstein metric $(ds)^2$ may be positive, negative or zero. These correspond to particles with speed less than c, greater than c, or equal to c, respectively. The second possibility is disallowed by special relativity.

The Einsteinian metric has the general form

$$(ds)^2 = \sum_i \sum_j g_{ij} du_i du_j \,.$$

For $(ds)^2 \geq 0$, the distance between two points along a given curve $u_k(s)$ is given by

$$s = \int ds = \int \sqrt{\sum_i \sum_j g_{ij} du_i du_j}$$

$$= \int \sqrt{\sum_i \sum_j g_{ij} \frac{du_i}{ds} \frac{du_j}{ds}} \, ds \,.$$

To find the geodesic (or shortest curve) between the two points, one applies calculus of variations to s to obtain the extremum. The Euler-Lagrange equation then gives the geodesic equation

$$\frac{d^2 u_k}{ds^2} + \sum_i \sum_j \Gamma_{ij}^k \frac{du_i}{ds} \frac{du_j}{ds} = 0 .$$

For flat spaces there exist global inertial coordinate systems. In such a coordinate system $\Gamma_{ij}^k = 0$ so that the geodesic equation becomes

$$\frac{d^2 u_k}{ds^2} = 0 .$$

This is just the equation of a straight line. Thus for flat spaces the geodesic law reduces to Galileo's law of inertia.

The geometric theory of gravity proposed by Einstein is known as the *general theory of relativity*. Once the metric of the space is found the geodesics are easily determined. In this space, material bodies simply follow the geodesic curves. The planet, for instance, follow geodesic paths in four-dimensional space time as they orbit around the sun.

How is the metric determined in a given distribution of matter or energy? The metric is governed by the ***Einstein field equation***

$$R_{ij} - \frac{1}{2} g_{ij} R = + \frac{8\pi G}{c^3} T_{ij} ,$$

where R_{ij} = Ricci tensor, R = scalar curvature, T_{ij} = energy-momentum tensor.

The Lagrangian-Hamiltonian formulation allows the use of generalized coordinates q_1, q_2, q_3, ..., q_n. Their corresponding generalized momenta are defined by

$$p_k = \frac{\partial L}{\partial \dot{q}_k} ,$$

where L is the Lagrangian of the system. For classical physics (mechanics, electromagnetism), q_k and p_k are numbers so that

$$q_k p_k - p_k q_k = 0 .$$

In quantum mechanics, the generalized coordinates and the generalized momentum are not necessarily numbers and

$$q_k p_k - p_k q_k = i\hbar ,$$

where $i = \sqrt{-1}$ and $\hbar$ = Planck's constant divided by 2π. Heisenberg found matrix representations of q_k and p_k, while Schroedinger obtained a calculus representation of these quantities. In the Schroedinger representation q_k is a number and

$$p_k = \frac{\hbar}{i}\frac{\partial}{\partial q_k} .$$

The energy E is represented by $i\hbar\frac{\partial}{\partial \tau}$. These quantities are supposed to operate or act on the wave function Ψ. Thus, for the harmonic oscillator whose energy is given by

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$= \frac{(mv)^2}{2m} + \frac{1}{2}kx^2$$

$$= \frac{p^2}{2m} + \frac{1}{2}kx^2, \qquad p = mv = \text{momentum},$$

the quantum-mechanical version in the Schroedinger representation is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi .$$

This is called the ***Schroedinger equation for the quantum harmonic oscillator***. Many problems in atomic, molecular and solid state physics are solved using the same prescription.

One cannot possibly hope to get a deep understanding of physics without mastering calculus. The preceding survey gives only a glimpse of a very limited area, yet, it already reveals the great power of calculus.