Some Applications of Calculus in Economics

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In economics, we talk a lot about curves: total cost curves, total revenue curves, total profit curves, total product curves, product transformation curves, etc., average curves, and marginal curves. And we use calculus to confirm certain propositions we arrive at, or to clarify some points or make precise definitions, or to solve some problems that can be formulated in mathematical expressions. The applications of calculus in economics are many, but we can mention only a few within the space allotted to us.

Marginal analysis and elasticity. Elementary economics tells us that given a *total cost function* $C = C(Q)$, where C denotes *total cost* and Q, the *output*, the ***marginal cost*** (MC) is defined as the change in total cost resulting from a one unit change in output, that is,

$$\text{MC} \equiv \frac{(C_2 - C_1)}{(Q_2 - Q_1)}.$$

It is understood that $(Q_2 - Q_1)$ is an extremely small change. For the case of the product that has discrete units, a change of one unit is the smallest change possible; but for the case of a product whose quantity is a continuous variable, the change in Q will refer to an infinitesimal change. In this latter case, the marginal cost can be measured by the slope of the total cost curve. But the slope of the total cost curve is nothing but the limit of the ratio

$$\frac{(C_2 - C_1)}{(Q_2 - Q_1)},$$

when $(Q_2 - Q_1)$ approaches zero. Thus, the concept of the slope of the total cost curve is merely the geometric counterpart of the concept of the derivative. Therefore,

$$\text{MC} \equiv \frac{dC}{dQ}.$$

This implies that given a total cost function, one can easily calculate the marginal cost at any level of output by just taking the derivative of the total cost function with respect to output to get the marginal cost function.

Other marginal concepts may be similarly defined precisely in the language of calculus. ***Marginal revenue*** (MR) may be concisely defined as

$$\text{MR} \equiv \frac{dR}{dQ},$$

where R is total revenue. ***Marginal profit*** (Mπ) may be defined as

$$\text{M}\pi \equiv \frac{d\pi}{dQ},$$

where π is total profit. The ***marginal product of labor*** (MP_L) may be defined as

$$\text{MP}_\text{L} \equiv \frac{dQ}{dL},$$

where L is labor (say, in man-hours); similarly *the* ***marginal product of capital*** (MP_K) may be defined as

$$\text{MP}_\text{K} \equiv \frac{dQ}{dK},$$

where K is capital (say, in machine–hours). All other "marginal" concepts may be defined in an analogous manner.

The *elasticity concepts* in economics so widely used in measuring the responsiveness of one variable to a change in the value of another variable are defined precisely in the language of calculus. The ***price-elasticity of demand for a product*** is defined as

$$\varepsilon = -\frac{dQ}{Q} \bigg/ \frac{dP}{P},$$

where ε is price-elasticity of demand, and P is the unit price of the product. The ***income-elasticity of the demand for money*** (or the preference to be liquid) is defined as

$$k = \frac{dL}{L} \bigg/ \frac{dY}{Y},$$

where k is the income-elasticity, L is liquidity preference (the demand for money), and Y is national income; similarly, the ***interest-elasticity of the demand for money*** is defined as

$$h = -\frac{dL}{L} \bigg/ \frac{di}{i},$$

where h is interest-elasticity, and i is the interest rate. All other elasticity concepts in economics are defined analogously.

Mathematics in economics. Calculus has also been used in economics to verify certain conclusions. One such important conclusion is the following:

8.1 PROPOSITION. *Total profit is maximum at the output rate where marginal revenue equals marginal cost.*

Discussion. Since $\pi = R - C$ (i.e., total profit = total revenue – total cost), if MR > MC at the current output rate, an additional unit that will be produced and sold will add more to total revenue than to total cost, and hence will contribute positively to total profit; while a reduction in output will decrease total revenue more than total cost, resulting in a reduction in total profit. On the other hand, if MC > MR at the current output rate, an additional unit produced and sold will add more to total cost than to total revenue, resulting in a reduction in total profit; while reducing production by one unit will reduce total cost more than total revenue, and hence will increase total profit. But at the output rate where MR = MC, the addition to total revenue is equal to the addition to total cost, hence, there is no more addition to total profit. Therefore, the output rate at this production level maximizes total profit.

In other words, if the total revenue function is $R = P \times Q = PQ$, where P is the price of the product, and the total cost function is $C = C(Q)$, then the total profit per period is

$$\pi = PQ - C(Q).$$

If π is a maximum, then

$$\frac{dPQ}{dQ} - \frac{dC(Q)}{dQ} = 0, \text{ or}$$

$$\frac{dPQ}{dQ} = \frac{dC(Q)}{dQ}, \text{ i.e., MR} = \text{MC}.$$

The calculus verifies that indeed MR = MC, if π is maximum. However, it also tells us MR = MC could also be at a minimum point (since the slope is also zero at the minimum point). To find out whether indeed we are at a maximum point, we take the second derivative and see whether this second derivative is negative (which is equivalent to checking whether the slope changes from zero to a downward slope as Q is increased).

Since the first derivative of π simplifies to $P - (dC(Q)/dQ) = 0$, the second derivative is therefore $-(d^2C(Q)/dQ^2)$, i.e., negative. Thus, the calculus confirms that indeed total profit is maximum at the output rate where MR = MC.

Another well-known result is the following:

8.2 PROPOSITION. *The marginal revenue product of an input x* (MRP_x) *is equal to the marginal product of input x multiplied by the marginal revenue*, i.e.,

$$\text{MRP}_x = \text{MP}_x \times \text{MR},$$

so that if the input is labor: $\text{MRP}_L = \text{MP}_L \times \text{MR}$.

Discussion. Given the total revenue function $R = f(L)$, where output Q is a function of labor input L, or $Q = g(L)$, we can find dR/dL which is the marginal revenue product of labor. By the chain rule, we have:

$$\frac{dR}{dL} = \frac{dQ}{dL}\frac{dQ}{dL}.$$

In economic terms, dR/dQ is the marginal revenue function, and dQ/dL is the marginal product of labor function. Similarly, dR/dL has the connotation of the marginal revenue product function. Thus, this result constitute the mathematical statement of the economic proposition that $\text{MRP}_L = \text{MR} \times \text{MP}_L$.

Yet another important conclusion in economics is the following:

8.3 PROPOSITION. *In order for a firm to maximize output given a production cost outlay, or to minimize production cost given an output quota, the firm should employ the combination of inputs in such a way that the marginal product per peso's worth of each input are all equal to each other.*

That is, if there are only two inputs, say, labor (L) and capital (K), the firm should employ the respective quantities of labor (in man-hours) and capital (in machine-hours) in such a way that

$$\frac{\text{MP}_L}{w} = \frac{\text{MP}_K}{i},$$

where w (wage rate) is the price of labor and i (interest rate) is the price of capital.

Discussion. This result can be shown graphically with the use of the ***isocost*** and ***isoquant*** curves, where the optimal solution is found at the input combination representing the point on the given isocost curve that is tangent to the highest isoquant, or the point on the given isoquant that is just tangent to an isocost line. The slope of the isocost is equal to the negative of the input price ratio w/i, while the slope of the isoquant is equal to the negative of the ratio of the marginal product of labor to the marginal product of capital (known as the ***marginal rate of technical substitution*** (MRTS) of labor for capital). Thus, $(w/i) = \text{MRTS}_{lk}$ at the point of tangency.

The production function considered here is $Q = Q(L,K)$. We can define two partial derivatives here: $\partial Q/\partial L$ and $\partial Q/\partial K$. The partial derivative $\partial Q/\partial L$ relates to the changes in output with respect to infinitesimal changes in the labor input while capital is held constant. Thus $\partial Q/\partial L$ symbolizes the marginal product of labor MP_L function. Similarly, the partial derivative $\partial Q/\partial K$ is the mathematical representation of the MP_K function. Hence, given the respective prices of the inputs and the production function, we can derive the marginal product functions by taking the partial derivatives with respect to the respective inputs and solve for L and K.

The ***marginal rate of technical substitution***, defined as $-dK/dL$, can be derived as follows: The total differential of the production function is

$$dQ = \frac{\partial f}{\partial L}dL + \frac{\partial f}{dK}dK.$$

Since output remains constant along an isoquant, i.e., $dQ = 0$ along a curve of constant Q (an isoquant). Thus

$$dQ = \frac{\partial f}{\partial L}dL + \frac{\partial f}{dK}dK = 0.$$

Hence,

$$\frac{\partial f}{\partial L}\Big/\frac{\partial f}{\partial K} = -\frac{\partial K}{\partial L}.$$

This says that the ratio of the marginal products is equal to the marginal rate of technical substitution. Therefore, $MP_L/MP_K = w/i$ at the optimal combination of labor and capital.

8.4 EXAMPLE. In 1953, agricultural economist Earl Heady [1] carried out an experiment to determine the effect of various quantities of nitrogen (N) and phosphate (P) on corn yield per acre, and found out that

$$Q = -5.682 - .316N - .417P + 6.351\sqrt{N} + 8.5155\sqrt{P} + .341\sqrt{PN}.$$

From this equation, it follows that

$$\mathrm{MP}_N = \partial Q/\partial N = -.316 + 3.1756\sqrt{1/N} + .1705\sqrt{P/N}, \text{ and}$$

$$\mathrm{MP}_P = \partial Q/\partial P = -.417 + 4.2578\sqrt{1/P} + .1705\sqrt{N/P}.$$

At the time of the experiment, the price of nitrogen was 18 cents per lb. and the price of phosphate was 12 cents per lb.. If a farm manager was thinking of spending $30 per acre on fertilizers, he could have determined how this expenditure should have been allocated between nitrogen and phosphate. The optimal N and P must be such that

$$\frac{\mathrm{MP}_N}{\mathrm{P}_N} = \frac{\mathrm{MP}_P}{\mathrm{P}_P},$$

and since he is going to spend $30,

$$.18N + .12P = 30.00.$$

Since we have two equations, the solutions P^* and N^* can be found ($N^* \approx 91$ lbs./acre, $P^* \approx 113.5$ lbs./acre).

Capital formation. Integrals are used in economic analysis in various ways. One application is in investment and capital formation. Capital formation is the process of adding to a given stock of capital. Regarding this process as continuous over time, we may express capital stock as a function of time, $K(t)$, and use the derivative dK/dt to denote the rate of capital formation. But the rate of capital formation at time t is identical with the rate of net–investment flow at time t, denoted by $I(t)$. Thus, capital stock K and net investment I are related by the following equations:

$$\frac{dK}{dt} = I(t) \text{ and } K(t) = \int I(t)dt = \int \frac{dK}{dt}dt = \int dK.$$

The first equation is an identity: it shows the synonymity between net investment and the increment of capital. Since $I(t)$ is the derivative of $K(t)$, it stands to reason that $K(t)$ is the integral or antiderivative of $I(t)$, as shown in the second equation. The transformation of the integrand is also easy to understand: The switch from I to dK/dt is by definition, and the next transformation is by cancellation of two identical differentials.

Present value of a cash flow. Consider a continuous revenue stream, like that of the toll fees from the operation of the South Superhighway in Metro Manila, at the rate of $R(t)$ pesos per year. This means that at $t = t + 1$ the rate of flow is $R(t)$ pesos per year, but at another point of time $t = t + 2$ the rate will be $R(t+2)$ pesos per year - with t taken as a continuous variable. If, at any point of time t we allow an infinitesimal time interval dt to pass, then the amount of revenue during the interval $[t, t+dt]$ can be written as $R(t)dt$. When discounted at the nominal interest rate of r per year, its present value should be $R(t)e^{-rt}dt$. If we let the problem be that of finding the total present value of a 25-year stream, the answer is to be found in the following definite integral:

$$PV = \int_0^{25} R(t)\, e^{-rt}\, dt.$$

Note that in the discrete case, the total present value of future amounts to be received over n years is the familiar sum

$$PV = \sum_{t=1}^{n} R_t (1+i)^{-t}.$$

Another simple application of integrals is in looking for the total function with the marginal function known. Given a total function, the process of differentiation can yield the marginal function. Being the opposite of differentiation, the process of integration should enable us to infer the total function from a given marginal function.

REFERENCE

[1] Heady, Earl, *An econometric investigation of the technology of agricultural production functions*, **Econometrica**, 1957.