APPLICATIONS OF
CALCULUS IN BUSINESS, MANAGEMENT, AND SOCIAL SCIENCES

Helen Moso Rara

Calculus is a branch of mathematics that helps us solve problems that involve variables. The present practice is to define a variable as an arbitrary element of a set. But when it is defined in this way, it losses all of the "motion" inherent in the word "variable". Nevertheless, it is easy to recapture the true meaning of the word merely by watching nature in action. For example, the distance of a car from some fixed point is a constant if the car is not moving, but the distance is a variable as soon as the car is in use. In the hands of a steady driver, the speed will be constant, but in the vast majority of cases the speed is a variable. The height of a bird in flight is a variable. The pressure of the gas in the cylinder of an automobile engine varies violently. The distance from the moon to the sun changes in a very complicated way.

In algebra the student learns how to solve a fixed equation for its fixed roots, or how to find the value of a certain fixed determinant or how to compute the probability of throwing a fixed number with a given set of dice. In trigonometry the student learns how to solve triangles that are fixed (at least while he is solving them), or to prove identities (which are also fixed). In calculus, however, we consider a variable and ask such questions as, "How fast is the quantity varying? What is the maximum value of the quantity? When is it decreasing, and when is it increasing?

The derivative. The concept of the derivative is probably the most useful concept of calculus. The derivative of a function tells us many things about the function. For instance, we can use the derivative to examine the rate of change of a function. Thus, suppose a certain function describes the concentration of an antibiotic in a patient's blood as a function of time, then the derivative of the function will tell us the rate at which the concentration of the antibiotics is changing with respect to time.

Derivative can also help us find the maximum and minimum values of a function. In this section we give examples and exercises on some applications of the derivative to business and management problems.

Rates of change. For any function f, the derivative f' gives the instantaneous rate of change of f with respect to its independent variable.

9.1 EXAMPLE. Students who take an intensive foreign language course in which only the language being taught is spoken to them typically understand

$$P = 0.005t^3 - 0.05t^2 + 2t$$

percent of what is being said to them by the end of day t of the course, $0 \leq t \leq 25$. Find the rate at which the comprehension of a typical student is changing 5 days, 10 days, and 15 days into the course.

Solution. The rate at which the comprehension of a typical student is changing at t days is

$$P'(t) = (0.005)(3)t^2 - (0.05)(2)t + 2.$$

Hence, at $t = 5$ days, 10 days, and 15 days,

$$P'(5) = (0.005)(3)(5)^2 - (0.05)(2)(5) + 2 = 1.875$$

$$P'(10) = (0.005)(3)(10)^2 - (0.05)(2)(10) + 2 = 2.5$$

$$P'(15) = (0.005)(3)(15)^2 - (0.05)(2)(15) + 2 = 3.875$$

Hence, the comprehension of a typical student is changing at 1.875%, 2.5%, and 3.875% at the end of 5 days, 10 days, and 15 days.

9.2 EXAMPLE. Alpha Company has studied the productivity of its employees and has been found that after t years of experience an employee's monthly productivity can be expected to be

$$f(t) = -2t^2 + 120t + 100$$

units. The derivative

$$f'(t) = -4t + 120$$

gives the rate of change of monthly productivity with respect to years of experience. For instance, when $t = 10$ we have $f'(10) = 80$. Thus employees with 10 years of experience are increasing their monthly productivity at a rate of 80 units per year. Similarly, since $f'(20) = 40$, employees with exactly 20 years of experience are also increasing their monthly productivity, but only at the rate of 40 units per year.

Note that $f'(30) = 0$, and hence employees with exactly 30 years of experience will be neither increasing nor decreasing their productivity. Since $f'(t) = -4t + 120$ is negative for $t > 30$, employee productivity decreases when $t > 30$. It follows that greatest productivity is reached when $t = 30$, and hence that maximum monthly productivity is $f(30) = 1900$ units.

Theory of the firm. The *theory of the firm* is concerned with questions such as these:

(1) How many units of its product must a firm make and sell in order to maximize its profit?

(2) What price should the firm charge for its product in order to attain its maximum profit?

(3) How will changing the product's price affect the firm's revenue?

Here we show how to use calculus to answer such questions. We do this by exploiting both viewpoints of the derivative: that it is the slope of the tangent line, and that it gives the rate of change with respect to the independent variable.

Marginality. If a firm produces and sell x units of its product then, as we have seen, it has a *cost function* C, a *revenue function* R, and a *profit function* P, where $P = R - C$. These functions need not be linear, but as in the linear case we can find the firm's *break-even quantities* by setting $R(x) = C(x)$ and solving for x, or, alternatively, by setting $P(x) = 0$ and solving for x.

9.3 EXAMPLE. Suppose Acme Company's cost and revenue functions are given by

$$C(x) = x^2 + 500{,}000 \quad \text{and} \quad R(x) = 1500x,$$

respectively. Then Acme's profit function is given by

$$P(x) = R(x) - C(x) = -x^2 + 1500x - 500{,}000.$$

Since

$$P(x) = -(x - 500)(x - 1000),$$

setting $P(x) = 0$ and solving for x yields $x = 500$ and $x = 1000$. Therefore, the firm's break-even quantities are 500 units and 1000 units. Note that Acme will make a profit if it makes and sells between 500 and 1000 units, but will suffer a loss if it makes and sells fewer than 500 or more than 1000 units.

The derivatives of the cost, revenue, and profit functions are called ***marginal*** functions. Thus C' is the *marginal cost function*, R' the *marginal revenue function*, and P' the *marginal profit function*. The economic interpretation of the marginal cost function is as follows: since the derivative $C'(a)$ is the instantaneous rate of change of the cost when $x = a$, $C'(a)$ represents the approximate additional cost of producing one more unit. In other words, $C'(a)$ is the approximate additional cost of producing the $(a + 1)$st unit. Similarly, $R'(a)$is the approximate additional revenue obtained from selling the $(a + 1)$st unit, and $P'(a)$ is the approximate addi-tional contribution to profit made by the $(a + 1)$st unit.

9.4 EXAMPLE. Let Acme Company's cost, revenue, and profit functions be as in Example 9.2. Then Acme's marginal cost, revenue, and profit functions are given by

$$C'(x)= 2x, \quad R'(x) = 1500, \quad \text{and} \quad P'(x) = -2x + 1500,$$

respectively. Since $C'(600) = 2(600) = 1200$, it will cost Acme approximately \$1200 to make the 601st unit. Similarly, since $R'(600) = 1500$, Acme's revenue from selling the 601st unit will be approximately \$1500. (Actually, in this example Acme's revenue from selling the 601st unit will be exactly \$1500.) Also, the approximate additional contribution to profit made by the 601st unit will be $R'(600) = \$300$.

Figure 1 below shows the graph of a typical profit function P. Notice that the maximum profit occurs when $x = a$ units are made and sold. Since the tangent line to P at $(a,P(a))$ is horizontal, its slope will be zero; hence, we must have $C'(a) = 0$. But since $P(x) = R(x) - C(x)$,

$$0 = P'(a) = R'(a) - C'(a),$$

and thus $R'(a) = C'(a)$. Thus, profit is maximized at the value of x for which

$$R'(x) = C'(x).$$

or, in the language of economics, at the quantity where marginal revenue equals marginal cost.

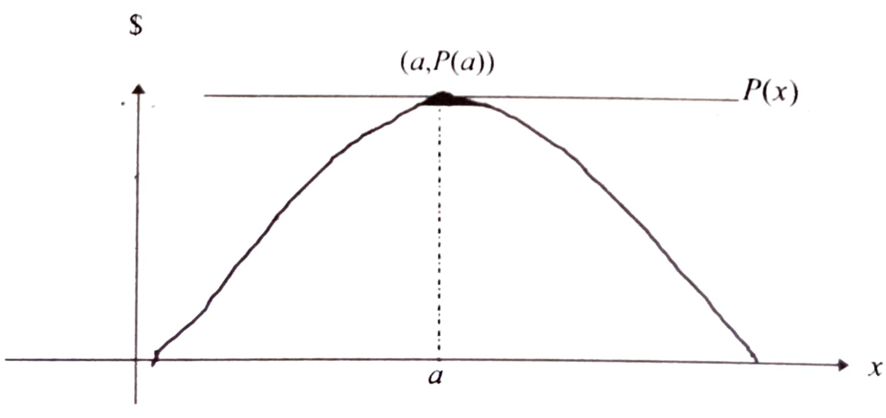

Figure 1

Profit maximization. Profit is maximized at the quantity for which marginal revenue equals marginal cost.

9.5 EXAMPLE. Consider Acme Company of Examples 2 and 3 again. If we set marginal revenue equal to marginal cost and solve for x, we have $C'(x) = R'(x)$, or $2x = 1500$. Thus, $x = 750$. Therefore, Acme's profit is maximized when it makes and sells 750 units, and its maximum profit is $P(750) = \$62,500$.

What if a firm has not yet determined the price it should charge for its product? For a single firm, the price that must be charged in order to obtain a given revenue depends on the demand for the product: if the demand is large, the firm can sell many units at a low price; if demand is small, it cannot sell very many units, and therefore must charge a higher price per unit in order to obtain the same revenue. If we let p denote the unit price of the product and the x the quantity demanded, then p depends on x by means of a demand function $p = d(x)$. Since revenue is quantity sold times unit price, it follows that the firm's revenue is given by

$$R(x) = xp = xd(x).$$

9.6 EXAMPLE. As in the previous examples, suppose Acme's Company's cost function is given by

$$C(x) = x^2 + 500,000,$$

but now suppose that the firm has not yet established a price for its product. Let the demand function for the product be defined by

$$p = 2100 - 0.5x.$$

Then Acme's revenue function is given by

$$R(x) = xp = x(2100 - 0.5x) = 2100x - 0.5x^2.$$

As before, we maximize profit by setting $C'(x) = R'(x)$ and solving for x: we have $2x = 2100 - x$ and hence $x = 700$.

Therefore, Acme will maximize its profit by making and selling 700 units. In order to sell 700 units, the firm must charge $p = 2100 - 0.5(700) = \$1750$.

Absolute maxima and minima. The largest value a function can assume over an interval is called its ***absolute maximum*** over the interval; the smallest value it can assume is its ***absolute minimum*** over the interval.

Inventory problem. An ***inventory problem*** is one concerned with minimizing the total cost of purchasing, ordering, and carrying a product in inventory. Total inventory cost is thus the sum of purchasing cost, ordering cost, and carrying cost, where

purchasing cost = (number of units purchased)(price per unit),

ordering cost = (number of orders placed)(cost of placing an order),

carrying cost = (average number of units in inventory) times (cost of carrying a unit in inventory).

We solve an inventory problem by finding the *optimal reorder quantity*, which is the number of units that should be ordered each time to minimize total inventory cost.

9.7 EXAMPLE. Suppose the Shoe Shop sells 2000 pairs of Bandolino each year. Each pair of Bandolino is purchased for \$10, and it costs \$100 to place an order for them. Also, it costs \$1.60 to carry a pair of shoes in inventory for 1 year. Let us find the optimal reorder quantity for this problem.

Solution. Let x denote the number of pairs of Bandolino that should be ordered each time an order is made. Since 2000 pair of Bandolino are required during the course of a year, they must be ordered $2000/x$ times per year. Suppose we assume that pairs of Bandolino are sold at a constant rate and that they sell out just as a new order arrives; then since each order consists of x pairs, the average number of pairs in inventory at any time will be $x/2$. Therefore, annual purchasing cost = (2000 pairs)($10 per pair) = $20,000.

$$\text{annual ordering cost} = \left(\frac{2000}{x}\text{ orders}\right)(\$100 \text{ per order}) = \frac{\$200{,}000}{x}.$$

$$\text{annual carrying cost} = \left(\frac{x}{2}\text{ pairs}\right)(\$1.60 \text{ per pair}) = \$0.80x.$$

Thus if $I(x)$ is the total annual inventory cost that results from ordering x pairs of Bandolino per order, then

$$I(x) = 20{,}000 + \frac{200{,}000}{x} + 0.80x,$$

where $x > 0$. We must minimize the inventory cost function I over the interval $(0,+\infty)$ Since

$$I'(x) = -\frac{200{,}000}{x^2} + 0.80,$$

setting $I'(x) = 0$ and solving for x yields $x = \pm\ 500$. Because x must be positive, we may discard the critical value -500. Note that we have placed a parenthesis "(" at $x = 0$ to denote that 0 is not included in our interval. It is clear that the minimum inventory cost occurs at $x = 500$. Therefore, the optimal reorder quantity is 500 pairs of Bandolino per order. This reorder quantity will result in 2000/500 = 4 orders per year, and the store's minimum annual inventory cost will be

$$I(500) = \$20{,}000 + \frac{\$200{,}000}{500} + \$0.80(500) = \$20{,}800.$$

Related rates. There is a common type of problem in which information about the rate of change of one variable with respect to another is known and it is required to find the rate of change of the second variable with respect to a third one. Such a problem is known as a *related rates problem.*

Related rates problems are solved with the aid of the chain rule. The key to solving related rates problems is to find an equation that relates the variable whose rate of change we want to the one whose rate of change we know; we can then differentiate this equation with the aid of the chain rule and solve for the unknown rate of change. Sometimes implicit differentiation is helpful.

9.8 EXAMPLE. Suppose the demand function for a product is given by $q = 10{,}000 - 400p$, where q is the quantity demanded and p is the unit price of the product. Suppose also that the price is given as a function of time by the equation $p = 5 + 2\sqrt{t}$, where t is time in months, with $t = 0$ representing the present. Let us find the rate at which demand for the product will be changing with respect to time 4 months from now.

We want to find the rate of change of quantity demanded q with respect to time t when $t = 4$; that is we want to find dq/dt when $t = 4$. We can easily find dp/dt from the equation $p = 5 + 2\sqrt{t}$. We have

$$\frac{dp}{dt} = \frac{1}{\sqrt{t}}.$$

The equation that relates q and p is $q = 10{,}000 - 400p$. Using the chain rule, we differentiate this equation with respect to t and obtain

$$\frac{dq}{dt} = -400\frac{dp}{dt} = -\frac{400}{\sqrt{t}}.$$

Therefore,

$$\left.\frac{dq}{dt}\right|_{x=4} = -\frac{400}{\sqrt{4}} = -200.$$

Thus 4 months from now demand for the product will be decreasing at the rate of 200 units per month.

Integration. In the applications of derivative we were concerned with the problem of finding the rate of change of a function. Often, however, it is not the function that is known and its rate of change that must be found, but rather the rate of change that is known and the function that must be found. For instance,

- a biologist might know the rate of population growth for a species and wish to find a function that describes the size of its population;

- a physicist might know the velocity of a particle and want to find a function that describes its position;

- a manufacturer might know the marginal cost function for a product and wish to find its cost function.

In cases like these, instead of starting with a function f and differentiating it to find its rate of change f', we start with a function f that describes a rate of change and *antidifferentiate* it to find a function F such that $F' = f$. Such a function F is called an *antiderivative* of f. The set of all antiderivatives of a function is known as the *indefinite integral* of the function. The *definite integral* of a function is a number that gives the total change of an antiderivative of the function over an interval.

Definite integral as total change. If f is a function defined on the interval $[a,b]$, then the ***definite integral of f over*** $[a,b]$, denoted by

$$\int_a^b f(x)\,dx,$$

is the total change in any antiderivative of $f(x)$ over the interval. Thus if $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

9.9 EXAMPLE. Research shows that people who learn a foreign language but never use it tend to forget their vocabulary at the rate of y percent per year, where

$$y = \frac{0.20}{t+1}.$$

Here t is time in years since the language was last used. How much of a language will be forgotten if it is not used for 10 years?

Solution. Using the definite integral, the percent of a language forgotten if it is not used for 10 years is

$$\int_0^{10} y(t)\,dt = \int_0^{10} \frac{0.20}{t+1}\,dt$$

By integration by substitution

$$\int_0^{10} \frac{0.20}{t+1}\,dt = (0.20)\ \ln\,(t+1)\Big|_0^{10} = .48\,.$$

Therefore 48% of the language is forgotten if it is not used for 10 years.

Producers' surplus and consumers' surplus. We show how to use the integral to calculate the quantities known in economics as *producers'* and *consumers' surpluses*, and how to apply the integral to the study of streams of income.

Let the equation $p = s(q)$ define a supply function for some commodity: here q is the number of units of the commodity supplied to the market, and p is the resulting unit price.

Similarly, let $p = d(q)$ define the demand function for the commodity. See Figure 2. Note that the market equilibrium quantity for the commodity is q_E, and its market equilibrium price is p_E.

In a free market the unit price of the commodity must stabilize at p_E. To see this, suppose the unit price is p_0. If $p_0 < p_E$, then $q_0 < q_E$ and there will be a shortage of the commodity, as shown in Figure 3. As consumers compete for the scarce commodity, they will bid up its price, and, therefore, the price will rise. On the other hand, if $p_0 > p_E$, then $q_0 > q_E$ and there will be a surplus of the commodity, as in Figure 4. In this case, producers must reduce the price in order to sell the commodity, so the price will fall. Thus, the only stable price in the long run is the equilibrium price p_E.

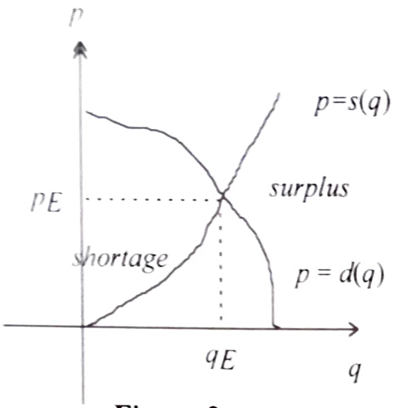

Figure 2

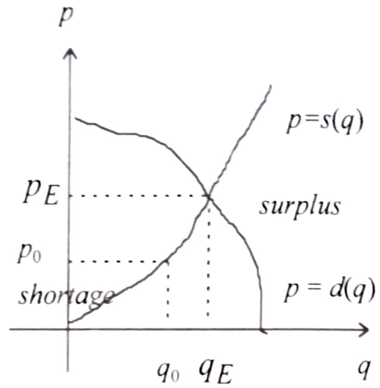

Figure 3

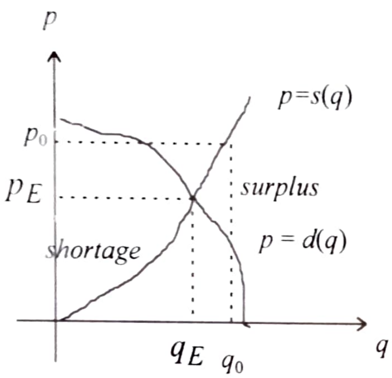

Figure 4

For the moment, let us consider only the supply curve. Some producers would be willing to supply the commodity at prices lower than the equilibrium price p_E; these producers gain when the price is at equilibrium. The total amount that producers as a group gain in this manner when the price is at equilibrium is called ***producers' surplus***. Analogous to producers' surplus is ***consumers' surplus.*** It is the total amount saved by consumers as a group when the price is at equilibrium.

By using of the integral as an accumulator, the producers' surplus is

$$\int_0^{q_E} \left[p_E - s(q)\right] dq$$

and the consumers' surplus is

$$\int_0^{q_E} \left[d(q) - p_E\right] dq .$$

9.10 EXAMPLE. Suppose the supply function for a commodity is defined by

$$s(q) = 0.009q^2$$

and its demand function by

$$d(q) = 30 - 0.003q^2.$$

Solving the equation $s(q) = d(q)$ for q yields

$$0.009q^2 = 30 - 0.003q^2, \quad \text{or}$$

$$q^2 = \frac{30}{0.012} = 2500\,.$$

Therefore, $q_E = 50$ and $p_E = s(50) = 22.50$. Thus, we have

$$\text{producers' surplus} = \int_0^{q_E} \left[p_E - s(q)\right] dq$$

$$= \int_0^{50} \left[22.50 - 0.009q^2\right] dq$$

$$= \$750,$$

and

$$\text{consumers' surplus} = \int_0^{q_E} \left[d(q) - p_E\right] dq$$

$$= \int_0^{50} \left[30 - 0.003q^2 - 22.50\right] dq$$

$$= \$250.$$

Income streams. Any firm continually receives income from its business activities. This is referred to as the firm's ***stream of income***

Future value of a stream of income. If income flows in continuously at a rate of $P(t)$ dollars per year and is reinvested at $100r$ percent per year compounded continuously, then the ***future value of the income stream*** at the end of b years is

$$\int_0^b P(t)e^{r(b-t)}dt\,.$$

9.11 EXAMPLE. The earnings of Zymax, Inc. are currently flowing into the firm at a constant rate of \$100,000 per year and are increasing at the rate of 6% per year. If we assume that earnings flow into the company continuously over the course of a year and are immediately reinvested at 6 percent per year compounded continuously, then we have $P(t)$ = \$100,000 and $r = 0.06$. Thus the value of Zymax's earnings stream over the next 4 years will be

$$\int_0^b P(t)e^{r(b-t)}dt = \int_0^4 100{,}000e^{0.06(4-t)}dt$$

$$= \int_0^4 100{,}000e^{0.24}e^{-0.06t}\,dt$$

$$= 100{,}000e^{0.24}\int_0^4 e^{-0.06t}\,dt$$

$$= 100{,}000e^{0.24}\left[\frac{e^{-0.06t}}{-0.06}\right]_0^4$$

$$= 100{,}000e^{0.24}\left[\frac{e^{-0.24}-1}{-0.06}\right]$$

$$= -\frac{100{,}000}{0.06}\left(1-e^{0.24}\right)$$

$$= \$452{,}082\,.$$

Present value of a stream of income. If a stream of income flows in continuously, arriving at the rate of $I(t)$ dollars per year at a time t, and if its future value is discounted at $100s\%$ per year compounded continuously, then the *present value of the stream of income* over the next b years is

$$\int_0^b I(t)e^{-st}\,dt$$

9.12 EXAMPLE. Suppose that each year Blink, Inc. earns \$100,000 in revenue. What is the present value of Blink's revenue stream over the next 4 years if Blink discounts future revenues at 8 percent per year compounded continuously? The answer is

$$\int_0^4 100{,}000e^{-0.08t}\,dt = -\frac{100{,}000}{0.08}\left(e^{-0.08t}\Big|_0^4\right)$$

$$= -\frac{100{,}000}{0.08}\left(e^{-0.32}-1\right)$$

$$= \$342{,}313.70\,.$$

9.13 EXAMPLE. Suppose Blink, Inc. of the previous example expects its revenue to continue to be \$100,000 per year forever. Then the present value of Blink's future revenue for all-time is

$$\int_0^{+\infty} 100{,}000e^{-0.08t}\,dt = \lim_{b\to+\infty}\int_0^b 100{,}000e^{-0.08t}\,dt$$

$$= -\frac{100{,}000}{0.08}\lim_{b\to+\infty}\left(e^{-0.08b}-1\right)$$

$$= -\,1{,}250{,}000(0-1)$$

$$= \$1{,}250{,}000.$$

The present value of an invested stream of income. If a stream of income is currently flowing in at a rate of P dollars per year, and if this income is invested at $100r\%$ per year compounded continuously and its future value is discounted at $100s\%$ per year compounded continuously, then the *present value of the income received* over the next b years is

$$\int_0^b Pe^{(r-s)t}\,dt .$$

9.14 EXAMPLE. Suppose that a firm's profits are currently flowing in at a rate of \$1 million per year and that it reinvests all profits. If the reinvested profits earn 8% per year compounded continuously and if the firm discounts future profits at 12% per year, then the present value of its profits over the next 5 years, in millions of dollars, will be

$$\int_0^5 1e^{(0.10-0.12)t}\,dt = \int_0^5 e^{-0.02t}\,dt$$

$$= -\left.\frac{e^{-0.02t}}{0.02}\right|_0^5$$

$$= -\frac{e^{-0.1}-1}{0.02} \approx 4.758,$$

or approximately \$4,758,000.

9.15 EXAMPLES. We round up a few additional examples from various disciplines for the readers to think about.

(1) If a blood vessel contracts, the velocity at which blood flows through it is affected. If the units of measurement are properly chosen, the average velocity of the blood flowing through an artery of radius x, where x is less than or equal to the normal radius r, is $V(x) = x^2(r - x)$. Using calculus, we can find the amount of contraction that maximizes the average velocity of the blood.

(2) The daily amount of smog produced by cars coming into a city is given by $y = 2 + 0.01x^{3/2}$, where y is smog concentration in parts per million and x is the number of cars in thousands. the number of cars coming into the city is increasing at the rate of 250 per day. Using derivatives, we can find the rate at which the smog concentration will be changing when a given number of cars are coming into the city.

(3) The pollutants in a pond are increasing at the rate of y grams per day, where $y = 10t + 3$ and t is in days with $t = 0$ representing the present. Find the total amount of pollutants that will accumulate in the pond over the next 30 days.

(4) A politician's campaign manager estimates that voters who support the politician can be registered at a rate of y individuals in t days, where

$$y = 2e^{0.25t}$$

with $t = 0$ being the present. Using integrals, we can determine the number of voters that can be registered over the next 60 days (It is almost 3.27M).

(5) **Survival-renewal functions.** Suppose that $p(t)$ is the size of a population at time t, $t \geq 0$. Let $f(t)$ be the proportion of the original population that survives at time t; the function f is called a ***survival function***. Let $g(t)$ denote the number of individuals added to the population at time t; the function g is called a ***renewal function***. We can use the integral as an accumulator to show that the size of the population at time $t = b$ is

$$p(b) = p(0)f(b) + \int_0^b g(t)f(b-t)\,dt\,.$$

REFERENCES

[1] Stancl, D. L. and Stancl, M. L., *Brief Calculus for Management and the Life and Social Sciences*, Richard D. Irwin, 1988.

[2] Goodman, A.W., *Calculus for the Social Sciences*, Macmillan, 1970