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The Equivalence of Real Number Principles

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In this note we shall examine eight properties of real numbers that are known to be equivalent. Partial proofs of their equivalence are found in some textbooks, e.g., see Mendelson [M, Ch. 5, Th. 5.9 & Th. 7.6], Buck [Bu, Sec. 1.7], and Parzynski and Zipse [PZ, p. 49]. Although these properties are standard fare in real analysis and advanced calculus courses, many are not aware that they are actually equivalent. In response to this deficiency, we shall give a complete proof of the said equivalence. The proof is broken into two series of propositions. Those that are readily available in standard textbooks are simply stated without proofs and the corresponding references are given.

1. Preliminaries

We begin with a list of the said properties.

(1) THE LEAST UPPER BOUND PROPERTY (LUB): *A nonempty set of real numbers which is bounded above has a least upper bound.*

(2) THE CAUCHY CONVERGENCE CRITERION (CCC): *Every Cauchy sequence of real numbers converges.*

(3) THE MONOTONE SEQUENCE PROPERTY (MSP): *Any bounded monotonic sequence of real numbers is convergent.*

(4) THE HEINE-BOREL PROPERTY (HB): *The closed interval $[a,b]$ is compact.*

(5) THE BOLZANO-WEIERSTRASS PROPERTY (BW): *Every bounded infinite set of real numbers has an accumulation point.*

(6) THE NESTED SETS PROPERTY (NSP): *Every nested sequence*

$$[a_1,b_1] \supseteq [a_2,b_2] \supseteq [a_3,b_3] \supseteq \cdots$$

of non-empty closed intervals has a non-empty intersection.

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(7) THOMSON'S LEMMA (TL): *A full cover* **C** *of the closed interval* $[a,b]$ *contains a partition of* $[a,b]$.

(8) COUSIN'S LEMMA (CL): *If* $\delta(x)$ *is a positive function defined on* **R**, *there exists a tagged partition* $D = \{(\mathbf{I}_k;\xi_k)\}$ *of* $[a,b]$ *such that for each k we shall have*

$$\xi_k \in \mathbf{I}_k \subseteq (\xi_k-\delta(\xi_k),\ \xi_k+\delta(\xi_k)).$$

Thomson's Lemma and Cousin's Lemma are late additions to this list. However, there is ample evidence, e.g., see [B1], [B2] and [C2], that they are of comparable importance to the others in the list. This is not surprising because, as we have mentioned earlier, they are actually equivalent to each other.

We begin our exposition with the Nested Sets Property, which has many important applications. For instance in [PZ, p. 49], it is used to prove the Bolzano-Weierstrass Theorem while in [B2, p. 328], it is used to prove Thomson's Lemma. In this paper we shall use it to prove the Least Upper Bound Property, which is usually called the Completeness Axiom in advanced calculus textbooks.

Proposition 1.0. *The Nested Sets Property implies the Least Upper Bound Property.*

Proof. Assume NSP and let **S** be a non-empty set of real numbers that is bounded above. We shall show that **S** has a least upper bound.

Without loss, we assume that $0 \in$ **S**. Let b be an upper bound for **S** such that $b > 0$. First we bisect $[0,b]$ into closed subintervals $[0,b/2]$ and $[b/2,b]$. Then we define $[a_1,b_1] = [0,b/2]$, if $b/2$ is an upper bound of **S**; otherwise, we take $[a_1,b_1] = [b/2,b]$.

Next we bisect $[a_1,b_1]$ and similarly define $[a_2,b_2]$. Continuing in this fashion indefinitely, we generate a nested sequence of closed subintervals of $[0,b]$:

$$[a_1,b_1] \supseteq [a_2,b_2] \supseteq [a_3,b_3] \supseteq \cdots .$$

By construction each interval $[a_n,b_n]$ contains elements of **S** and each b_n is an upper bound of **S**. By the Nested Sets Property, there is an $x \in [0,b]$ such that $x \in \bigcap_n[a_n,b_n]$. Now since the length of the nth subinterval $(= (b/2^n))$ tends to 0, we have $\bigcap_n[a_n,b_n] = \{x\}$.

Clearly the sequence (a_n) is increasing and the sequence (b_n) is decreasing. Moreover, they tend to the same limit x.

Now if some a_n is an upper bound for **S**, then it is clear that $a_n \in$ **S** because $[a_n,b_n]\cap$**S** $\neq \varnothing$. In this case we must have $a_n = \sup$ **S** and we are done. Therefore, we shall assume that no a_n is an upper bound for **S**.

Let **U** be the set of all upper bounds for **S**. Claim: **U** $= [x,\infty)$, and, therefore, $\sup$ **S** $= x$.

If $t < x$, then $t < a_m$ for some m. Hence, t is not an upper bound for **S** because a_m, by assumption, is not an upper bound for **S**. Thus $\mathbf{U} \subseteq [x,\infty)$.

On the other hand, if $u > x$, then $u > b_m$ for some m. Hence, u is an upper bound for **S** because b_m is. Consequently, no element of **S** is greater than x, This implies that x is also an upper bound for **S**. Therefore, $[x,\infty) \subseteq \mathbf{U}$. Thus, $\mathbf{U} = [x,\infty)$. □

To complete the proof of equivalence, we shall establish in the next two sections two series of implications. These are

(1.1) LUB ⇒ CCC ⇒ MSP ⇒ NSP ⇒ LUB, and

(1.2) HB ⇒ TL ⇒ CL ⇒ BW ⇒ NSP ⇒ LUB ⇒ HB.

2. Properties of Real Sequences

The next propositions aim to establish the first three implications in (1.1).

Proposition 2.1. *The Least Upper Bound Property implies the Cauchy Convergence Criterion.*

Proof. We shall present a version of the proof given in [C1]. Assume LUB and let (x_k) be a Cauchy sequence of real numbers. Let **S** be the collection of all real numbers x such that the interval $(-\infty,x]$ contains at most finitely many terms of (x_k). Observe that:

(*i*) if x is in **S**, then $(-\infty,x] \subseteq \mathbf{S}$ and

(*ii*) if $x \notin \mathbf{S}$, then $\mathbf{S} \subseteq (-\infty,x]$; thus x is an upper bound for **S**.

Now let $\varepsilon > 0$. Because the sequence (x_k) is Cauchy, there is an integer N such that if $j, m \geq N$, then $x_j - \varepsilon < x_m < x_j + \varepsilon$.

Hence, if $j \geq N$, then, by (*i*) and (*ii*), $(x_j - \varepsilon) \in \mathbf{S}$ and $(x_j + \varepsilon)$ is an upper bound for **S**. Hence, **S** is a non-empty set of real numbers, which is bounded above. Thus, by the Least Upper Bound Property, **S** has a least upper bound, say σ. We shall show that $\lim x_k = \sigma$.

As observed above, for a given $\varepsilon > 0$, there is an integer N such that if $j \geq N$, then $(x_j - \varepsilon) \in \mathbf{S}$ and $(x_j + \varepsilon)$ is an upper bound for **S**. Therefore, we must have $x_j - \varepsilon \leq \sigma \leq x_j + \varepsilon$, whenever $j \geq N$. Hence, $|x_j - \sigma| \leq \varepsilon$ for all $j \geq N$, which implies that $\lim x_k = \sigma$. □

Proposition 2.2. *The Cauchy Convergence Criterion implies the Monotone Sequence Property.*

Proof. Let (a_n) be a bounded increasing sequence of real numbers. We shall show that it converges by showing that it is a Cauchy sequence.

Let $\varepsilon > 0$ and assume without loss that $0 \le a_n \le b$ for all n. By the Archimedean Principle, there is a positive integer j such that $j\varepsilon > b$. Hence, there is a least positive integer m such that $a_n \le m\varepsilon$ for all n. By definition of m, we have $(m-1)\varepsilon < a_N$, for some $N \in \mathbf{N}$. Since (a_n) is increasing, it follows that for all $n, k \ge N$, $(m-1)\varepsilon < a_k, a_n \le m\varepsilon$.

Therefore, $|a_k - a_n| < \varepsilon$. for all $n, k \ge N$. This proves that (a_n) is a Cauchy sequence and is therefore convergent. □

Proposition 2.3. *The Monotone Sequence Property implies the Nested Sets Property.*

Proof. Assume MSP and suppose that $[a_1,b_1] \supseteq [a_2,b_2] \supseteq [a_3,b_3] \supseteq \cdots$ is a nested sequence of non-empty closed intervals. Let us show that $\bigcap_n[a_n,b_n] \ne \varnothing$.

The sequence (a_n) of lower endpoints is bounded and increasing. Hence, by assumption, it converges, say, to a limit α. Moreover since each b_n is an upper bound for all the sequence (a_n), we must have $a_n \le \alpha \le b_n$ for each n. Hence the set $\bigcap_n[a_n,b_n]$ contains α and is therefore non-empty. □

Proposition 2.4. LUB $\Rightarrow$ CCC $\Rightarrow$ MSP $\Rightarrow$ NSP $\Rightarrow$ LUB.

3. Topological Properties

Throughout this section we fix a closed interval $[a,b]$. Moreover we shall assume that **I**, with or without subscripts, denotes a closed subinterval of $[a,b]$ and $|\mathbf{I}|$ denotes the length of **I**.

DEFINITION 3.0. Let $X \subseteq [a,b]$. A collection **C** of closed subintervals of $[a,b]$ is a ***full cover*** of X, if to each x in X, there is a positive number $\delta(x)$ such that the **FC–*condition*** for $[a,b]$ holds:

> *Every closed subinterval* **I** *of* $[a,b]$ *that contains x and has length* $|\mathbf{I}| < \delta(x)$ *belongs to* **C**.

Intuitively, a full cover **C** of X, like a Vitali cover, includes all arbitrarily small closed subintervals of $[a,b]$ that meet X. The importance of full covers stems from the observation that a full cover of the interval $[a,b]$ includes a partition of $[a,b]$.

Recall that a ***partition*** of the closed interval $[a,b]$ is a finite, increasing collection $\{ x_0 = a < x_1 < x_2 < \cdots < x_n = b \}$ of points of $[a,b]$. It divides $[a,b]$ into closed non-overlapping subintervals $[x_{i-1},x_i]$. A ***tagged partition*** of the interval $[a,b]$ is a finite collection of ***interval-point pairs*** $(\mathbf{I}_k;\xi_k)$ satisfying certain conditions and such that the collection $\{\mathbf{I}_k : k = 1, 2, \ldots, n\}$ is a partition of $[a,b]$.

Theorem 3.1 (Thomson's Lemma). *If* **C** *is a full cover of* $[a,b]$, *then* **C** *contains a partition of* $[a,b]$.

Proof. Thomson proved this lemma using the Nested Sets Property and his proof is reproduced in [B2; p. 328]. We shall prove the lemma using the Heine-Borel Property.

Let **C** be a full cover of $[a,b]$. Then there is a positive function $\delta(x)$ defined on $[a,b]$ satisfying the FC-condition. Now define

$$\mathbf{L} = \{(x - \frac{\delta(x)}{2}, x + \frac{\delta(x)}{2}) : x \in [a,b]\}.$$

Clearly **L** is an open cover of $[a,b]$. By the Heine-Borel Property, there exist finitely many points $x_1 < x_2 < x_3 < \cdots < x_m$ in (a,b) such that the open intervals $\mathbf{J}_k = (x_k - \delta(x_k)/2, x_k + \delta(x_k)/2)$, where $k = 1, 2, ..., m$, cover $[a,b]$. (Without loss, we shall assume that no $\mathbf{J}_k$ contains another $\mathbf{J}_{k^\bullet}$.) Now we choose $t_0 = a$ and $t_m = b$. For $k = 1, 2, ..., (m-1)$, we choose a t_k in $\mathbf{J}_k \cap \mathbf{J}_{k+1}$ such that $x_k < t_k < x_{k+1}$. Then each x_k is in the subinterval $[t_{k-1}, t_k]$, and the length of this subinterval, which is a subset of $\mathbf{J}_k$, is less than $\delta(x_k)$. Hence, each $[t_{k-1}, t_k]$ is in the collection **C**. Since

$$\{ t_0 = a < t_1 < t_2 < \cdots < t_m = b \}$$

is a partition of $[a,b]$, this proves the lemma. □

Corollary 3.1.1. *The Heine-Borel Theorem implies Thomson's Lemma.*

Theorem 3.2 (Cousin's Lemma). *If* $\delta(x)$ *is a positive function defined on* **R**, *there exists a tagged partition* $D = \{(\mathbf{I}_k;\xi_k)\}$ *of* $[a,b]$ *such that for each k, we have* $\xi \in \mathbf{I} \subseteq (\xi_k - \delta(\xi_k), \xi_k + \delta(\xi_k))$.

Proof. Let δ be a positive function defined on **R**. Consider the collection **C** of closed subintervals **I** of $[a,b]$ such that for some x in $[a,b]$, we have $x \in \mathbf{I} \subseteq (x - \delta(x), x + \delta(x))$. Then the collection **C** is clearly a full cover of $[a,b]$. Therefore, by Thomson's Lemma, **C** contains a partition of $[a,b]$. □

Corollary 3.2.1. *Thomson's Lemma* (TL) *implies Cousin's Lemma* (CL).

As a simple application of Cousin's Lemma we shall present the proof of the Bolzano-Weierstrass Theorem given in [B1; p. 452].

Theorem 3.3 (Bolzano–Weierstrass). *If* **S** *is a bounded, infinite set of real numbers, then* **S** *has an accumulation point.*

Proof. Suppose that $\mathbf{S} \subseteq [a,b]$, for some $a, b \in \mathbf{R}$. We shall prove the contrapositive: *If the bounded set* **S** *has no accumulation point, then it is finite.*

Let $x \in [a,b]$. Then x is not an accumulation point of **S**. This implies that there is a $\delta(x) > 0$ such that $S \cap (x-\delta(x), x+\delta(x))$ is finite. If $x \notin [a,b]$, we define $\delta(x) = 1$. By Cousin's Lemma, there is a tagged partition $D = \{(\mathbf{I}_k, \xi_k)\}$ such that $\xi_k \in \mathbf{I}_k \subseteq (\xi_k-\delta(\xi_k), \xi_k+\delta(\xi_k))$ for each k. Since $S \cap \mathbf{I}_k \subseteq S \cap (\xi_k-\delta(\xi_k), \xi_k+\delta(\xi_k))$ is finite for each k, it follows that $\mathbf{S} = S \cap [a,b] = S \cap (\cup_k \mathbf{I}_k) = \cup_k (S \cap \mathbf{I}_k)$ is finite, being a finite union of finite sets. □

Corollary 3.3.1. *Cousin's Lemma implies the Bolzano-Weierstrass Property.*

Proposition 3.4. *The Bolzano-Weierstrass Property implies the Nested Sets Property.*

Proposition 3.5. *The Least Upper Bound Property implies the Heine-Borel Property*

The proofs of Propositions 3.4 and 3.5 can be found in [A; p. 56] and [Bu; Ch. 1, Th. 24]. Now since the Nested Sets Property implies the Least Upper Bound Property (Proposition 1.0) we have completed the proof of (1.2).

Proposition 3.6. HB $\Rightarrow$ TL $\Rightarrow$ CL $\Rightarrow$ BW $\Rightarrow$ NSP $\Rightarrow$ LUB $\Rightarrow$ HB.

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