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On the Dimensions of a Graph And Its Complement

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Abstract

An initial investigation on the dimension of the complement of a graph was done by Gervacio and Raposa in [4]. *In particular, some results and a question regarding the sum* $dim(G) + dim(\overline{G})$, *where* $\overline{G}$ *is the complement of a graph G, were given.*

In [2], *the authors proved a result which gives the exact dimension of any graph that is obtained from the complete graph by deletion of some special edges. The present paper gives upper bounds for* $dim(\overline{G})$ *as well as the sum* $dim(G) + dim(\overline{G})$ *for some special graphs G.*

1. Preliminary Concepts and Known Results

The graphs considered here are simple graphs, i.e., they are finite, loopless, and without multiple edges. We denote by $V(G)$, $E(G)$ and $\overline{G}$ the vertex-set, the edge-set, and the complement, respectively, of a graph G. For some terms, concepts, and graph operations whose definitions are assumed and are not given here, the reader may refer to [6].

The ***Euclidean n-space*** $\mathbf{R}^n$ is the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers x_i. The elements of $\mathbf{R}^n$ are called ***points***. If p and q are two points in $\mathbf{R}^n$, the ***Euclidean distance*** between them is denoted by $|p - q|$.

For convenience, we define the ***Euclidean*** 0-***space*** $\mathbf{R}^0$ to be the space containing only the zero point.

Definition 1.1. A ***unit representation*** of a graph G in the Euclidean n-space $\mathbf{R}^n$ is a one-to-one mapping $\phi : V(G) \rightarrow \mathbf{R}^n$ such that $|\phi(x) - \phi(y)| = 1$ whenever $[x,y]$ is in $E(G)$.

Definition 1.2. A graph G is called a ***unit graph*** in $\mathbf{R}^n$ if it has a unit representation in $\mathbf{R}^n$.

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The following remarks are easy to verify.

Remark 1.3. If G is a unit graph in $\mathbf{R}^n$, then so is it in $\mathbf{R}^m$ for $m > n$.

Remark 1.4. If G is a graph of order n, then G is a unit graph in $\mathbf{R}^n$.

Definition 1.5. The ***Euclidean dimension***, or simply ***dimension***, of a graph G, denoted by $dim(G)$, is the smallest nonnegative integer n for which G is a unit graph in $\mathbf{R}^n$.

The following remarks are proved in [7].

Remark 1.6. If H is a subgraph of G, then $dim(H) \leq dim(G)$.

Remark 1.7. If H and G are isomorphic graphs, then $dim(H) = dim(G)$.

Definition 1.8. Let G be graph of order $n \geq 3$, and Ω a family of proper subgraphs of G. We say that Ω is an ***independent set*** if no two distinct subgraphs in Ω have a common vertex.

Definition 1.9. The ***independence number*** of a graph G, denoted by $\beta(G)$, is the largest cardinality of an independent set Ω consisting of subgraphs of G isomorphic to K_2.

Definition 1.10. The ***triangle independence number*** of a graph G, denoted by $t(G)$, is the largest cardinality of an independent set Ω consisting of subgraphs of G isomorphic to K_3.

The following result is found in [1] and [3].

Lemma 1.11. *For any n*, $dim(K_n) = n - 1$.

Definition 1.12. Let K_n be a complete graph of order $n \geq 3$, and Ω an independent family of complete proper subgraphs of K_n each of order at least 2. The ***graph G obtained from K_n by deleting the family*** Ω, denoted by $K_n - \Omega$, is the graph of order n such that $[x,y]$ is in $E(G)$ if and only if $[x,y]$ is not an edge in any subgraph in Ω. If the elements of Ω are all of order 2, then we sometimes say that $K_n - \Omega$ is a graph obtained from K_n by deleting independent edges.

2. Results

Lemma 2.1. For $n \geq 3$,

$$\beta(P_n) = \beta(C_n) = \beta(W_n) = \beta(K_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd (for } W_n\text{),} \\ \frac{n-1}{2} & \text{if } n \text{ is odd (otherwise).} \end{cases}$$

Proof. Assume that $P_n = [1, 2, \ldots, n]$. Let $e_1 = [1,2]$. Then $P_n - e_1 \cong K_1 \dot{\cup} P_{n-1}$, where $K_1 \dot{\cup} P_{n-1}$ the disjoint union of K_1 and $P_{n-1} = [2, 3, \ldots, n]$. Any edge in P_{n-1} except [2,3] will form an independent set with e_1. So, take $e_2 = [3,4]$. Then $P_n - \{e_1, e_2\} \cong K_1 \dot{\cup} \overline{K_2} \dot{\cup} P_{n-3}$, where $P_{n-3} = [4, 5, \ldots, n]$. Next, take $e_3 = [5,6]$ in P_{n-3}. Then $\{e_1, e_2, e_3\}$ is independent and

$$P_n - \{e_1, e_2, e_3\} \cong K_1 \dot{\cup} 2\overline{K_2} \dot{\cup} P_{n-5}.$$

Continuing in this manner, we see that if $S_k = \{e_1, e_2, \ldots, e_k\}$ is an independent set, then $P_n - S_k \cong K_1 \dot{\cup} (k-1)\overline{K_2} \dot{\cup} P_{n-(2k-1)}$. This process of generating a bigger independent set terminates when $n - (2k - 1)$, which occurs if n is even, or when $n - (2k - 1) = 2$, which occurs if n is odd. The corresponding k which is $\beta(P_n)$ is seen to be $\beta(K_n)$.

The proofs of the remaining parts are virtually similar. □

In [2], the following result is obtained.

Theorem 2.2. *Let $n \geq 3$ and Ω a nonempty independent family of complete proper subgraphs of K_n each of order at least 2. If $G = K_n - \Omega$, then*

$$dim(G) = \begin{cases} n-2 & \text{if } \alpha = 1, \beta = 0, \\ n-r+1 & \text{if } \alpha = 0, \beta = 1, \\ n-r-\alpha+2\beta & \text{if } \alpha+\beta \geq 2, \end{cases}$$

where $\alpha = |V|$, $\beta = |W|$, and $r = (W)\Sigma p$ for $V = \{ K_p \in \Omega : p = 2 \}$ and $W = \Omega - V$.

In [2], the authors gave two direct consequences of the above theorem. The following is the general version of those results.

Corollary 2.3. *Let $n \geq 3$ and Ω a nonempty independent family of complete proper subgraphs of K_n such that each subgraph is of order m, where $m \geq 2$. If $|\Omega| = k$, then*

$$dim(K_n - \Omega) = \begin{cases} n-2 & \text{if } k=1, m=2, \\ n-m+1 & \text{if } k=1, m>2, \\ n-k & \text{if } k>1, m=2, \\ n-km+k & \text{if } k>1, m>2. \end{cases}$$

Proof. The cases where $m = 2$ and $m = 3$ are given in [2]. So, suppose that $m > 3$. If $k = 1$, then using the notations given in the preceding theorem, it follows that $\alpha = 0$, $\beta = 1$, and $r = m$. Thus, by Theorem 2.2, $dim(K_n - \Omega) = n - m + 1$. If $k > 1$, then by Theorem 2.2 (with $\alpha = 0$, $\beta = k$, and $r = km$) we have $dim(K_n - \Omega) = n - km + k = n - (m - 1)k$. □

Theorem 2.4. *Let G be a graph of order n. If G is unit graph in the plane, then $dim(G) + dim(\overline{G}) \leq n$.*

Proof. The inequality is true for $n = 1$ and 2. So, we assume that $n \geq 3$. If $E(G) = \varnothing$, then $\overline{G} = K_n$ and $dim(G) + dim(\overline{G}) \leq 1 + (n - 1) = n$. If $E(G) = \varnothing$, then $\overline{G}$ is a subgraph of a graph obtained from K_n by deleting an edge. By Corollary 2.3 and Remark 1.6, it follows that $dim(\overline{G}) \leq n - 2$. Accordingly, $dim(G) + dim(\overline{G}) \leq n$. □

Theorem 2.5. *Let $n \geq 3$ and $1 \leq k \leq \beta(K_n)$. If G is a graph of order n, $dim(G) \leq k$, and G has k independent edges, then $dim(G) + dim(\overline{G}) \leq n$.*

Proof. Clearly, the inequality holds if $E(G) = \varnothing$. So, suppose that $E(G) \neq \varnothing$. Because G contains k independent edges, $\overline{G}$ is a subgraph of a graph obtained from K_n by deleting k independent edges. By Corollary 2.3 and Remark 1.6, $dim(\overline{G}) \leq n - k$. Therefore, $dim(G) + dim(\overline{G}) \leq n$. □

Observe that Theorem 2.5 is useful in finding an upper bound for the dimension of the complement of some non-unit graphs in $\mathbf{R}^2$. To see this, consider the following.

Example 2.6. Let $G = F_7$, the fan of order 8. It is given in [1] that $dim(F_7) = 3$. Since F_7 has three independent edges, Theorem 2.5 says that $dim(\overline{F_7}) \leq 5$.

Next, we give upper bounds for the dimension of the complement of some special graphs.

Theorem 2.7. [1] *Let* $n \geq 1$. *Then*

$$dim(P_n) = \begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{if } n > 1, \end{cases}$$

$$dim(F_n) = \begin{cases} 2 & \text{if } n = 2,\ 3,\ 4,\ 5,\ 6 \\ 3 & \text{if } n > 6. \end{cases}$$

If $n \geq 3$, *then*

$$dim(C_n) = 2 \quad \text{and}$$

$$dim(W_n) = \begin{cases} 2 & \text{if } n = 6, \\ 3 & \text{if } n \neq 6. \end{cases}$$

Lemma 2.8. *Let* $n \geq 1$. *Then*

$$dim(\overline{P_n}) = \begin{cases} 0 & \text{if } n = 1, 2 \\ 1 & \text{if } n = 3, 4 \end{cases}$$

and

$$dim(\overline{P_n}) \leq \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. It is easy to see that the result holds for n = 1, 2, 3, and 4. Assume that $n \geq 5$. By Lemma 2.1, it follows that $\overline{P_n}$ is a subgraph of a graph obtained from K_n by deleting $\beta(K_n) = \beta(P_n)$ independent edges. Thus, by Remark 1.6 and Corollary 2.3, we get the desired result. □

Theorem 2.9. *For any* $n \geq 1$,

$$dim(P_n) + dim(\overline{P_n}) = \begin{cases} 0 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 2 & \text{if } n = 3, 4 \end{cases}$$

and

$$dim(P_n) + dim(\overline{P_n}) \leq \begin{cases} \frac{n+2}{2} & \text{if } n \text{ is even,} \\ \frac{n+3}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. This follows from Theorem 2.7 and Lemma 2.8. □

Theorem 2.10. *If* $n \geq 2$, *then*

$$dim(F_n) + dim(\overline{F_n}) \leq \begin{cases} \frac{n+4}{2} & \text{if } n \leq 6,\ n \text{ even,} \\ \frac{n+5}{2} & \text{if } n \leq 6,\ n \text{ odd,} \\ \frac{n+6}{2} & \text{if } n > 6,\ n \text{ even,} \\ \frac{n+7}{2} & \text{if } n > 6,\ n \text{ odd.} \end{cases}$$

Proof. For all $n \geq 2$, since $F_n = K_1 + P_n$, we have $dim(\overline{F_n}) = dim(\overline{P_n})$. Hence, the result is immediate from Theorem 2.7 and Lemma 2.8. □

Lemma 2.11. *If* $n \geq 3$, *then*

$$dim(\overline{C_n}) \leq \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By Lemma 2.1, C_n has $\beta(K_n)$ independent edges. Thus, $\overline{C_n}$ is a subgraph of a graph obtained from K_n by deleting $\beta(K_n)$ edges. By Corollary 2.3 ($m = 2$ and $k = \beta(K_n)$) and Remark 1.6, we have the desired result. □

Theorem 2.12. *For* $n \geq 3$,

$$dim(C_n) + dim(\overline{C_n}) \leq \begin{cases} \frac{n+4}{2} & \text{if } n \text{ is even,} \\ \frac{n+5}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. This follows from Theorem 2.7 and Lemma 2.11. □

Corollary 2.13. *If G is a Hamiltonian graph of order $n \geq 3$, then*

$$dim(\overline{G}) \leq \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Since G is Hamiltonian, it contains a cycle of order n. It follows that $dim(\overline{G}) \leq dim(\overline{C_n})$. The result is now immediate from Lemma 2.11. □

Corollary 2.14. *If G is a Hamiltonian graph of order $n \geq 3$, then*

$$dim(G) + dim(\overline{G}) \leq \begin{cases} \frac{3n-4}{2} & \text{if } n \text{ is even,} \\ \frac{3n-3}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If $G = K_n$, then $dim(G) + dim(\overline{G}) \leq n$ and so, the result holds. If $G \neq K_n$, then $\overline{G} \neq \varnothing$. This means that G is a subgraph of some graph obtained from K_n by deleting an edge. By Corollary 2.3 and Remark 1.6, $dim(G) \leq n - 2$. Combining this with Corollary 2.13, we see that the desired inequality holds. □

Observe that in Corollary 2.13 and Corollary 2.14, nothing has been said about the dimension of the Hamiltonian graph G in the hypotheses. The following result gives a better upper bound if an additional assumption is imposed on the dimension of G.

Corollary 2.15. *If G is a Hamiltonian graph of order $n \geq 3$ and $dim(G) \leq \beta(K_n)$, then $dim(G) + dim(\overline{G}) \leq n$.*

Proof. This follows from Lemma 2.1 and Corollary 2.13. Notice that the result also follows from Theorem 2.5 because according to Lemma 2.1, G has $\beta(K_n)$ independent edges. □

Lemma 2.16. *If $n \geq 3$, then*

$$dim(\overline{W_n}) \leq \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Since $W_n = K_1 + C_n$ for all $n \geq 3$, it follows that $dim(\overline{W_n}) = dim(\overline{C_n})$. Thus, the result follows from Lemma 2.11. □

Theorem 2.17. *If* $n \geq 3$, *then*

$$dim(W_n) + dim(\overline{W_n}) \leq \begin{cases} 5 & \text{if } n = 6, \\ \dfrac{n+6}{2} & \text{if } n \text{ is even}, \\ \dfrac{n+7}{2} & \text{if } n \text{ is odd}. \end{cases}$$

Proof. This follows from Theorem 2.7 and Lemma 2.16. □

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