The Mindanao Forum, Vol. XIV, No. 2 (December 1999)

A Vector-Valued Nonlinear Integral

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N. Dinculeanu pointed out that every new type of transformation needs the development of a new integral for its representation (see [5]). In fact, this approach was adopted by Riesz in proving the classical Riesz representation theorem. He used the Riemann-Stieltjes integral for the purpose of representing a functional.

For functionals and operators which are not necessarily linear, it is quite natural to develop nonlinear integrals for their representation. Drewnowski and Orlicz [6], for example, studied orthogonally additive functionals and gave a nonlinear integral representation of these nonlinear functionals.

Lee in [7] also developed a nonlinear integral of Henstock type. Using this integral, Chew in [4] was able to represent boundedly continuous and orthogonally additive functionals on the Denjoy space.

1. Preliminaries

In this paper, we develop the vector version of Lee's nonlinear Henstock integral. This vector formulation was used earlier in [1] to represent orthogonally additive operators on the space of regulated functions.

Definition 1. Let $\phi : X \times \vartheta \rightarrow Y$ be a function, where X and Y are real Banach spaces and ϑ is the family of all intervals in $[a,b]$. A function $f : [a,b] \rightarrow X$ is said to be ***ϕ-integrable*** on $[a,b]$ if there exists a vector $A \in Y$ such that for every $\varepsilon > 0$ there exists a $\delta(\xi) > 0$ defined on $[a,b]$ such that for every δ-fine division D = $\{([u,v];\xi)\}$ of $[a,b]$, we have

$$\left\| (D)\sum \phi(f(\xi),[u,v]) - A \right\| < \varepsilon ,$$

where $(D)\Sum$ is used to denote that the sum is over all interval-point pairs $([u,v]:\xi)$ in D. In this case, the ϕ-integral of f on $[a,b]$ is A. For convenience, we write

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$$\int_a^b \phi(f(t), dt) = A.$$

Using a standard argument, it can be shown that the ϕ-integral of a function, if it exists, is uniquely determined. Also, it is worth noting that if $X = Y$ and $\phi(x,I) = x(v - u)$, where u and v are the endpoints of the interval I, then Definition 1 reduces to the Henstock-Bochner integral (see [2]). Further, since $\phi(x,I)$ is not necessarily linear in x, the integral defined earlier is a nonlinear integral.

In order that the integral has some reasonable properties, we assume throughout that the function ϕ satisfies the following conditions:

(N1) $\phi(\theta_X,I) = \theta_Y$, where θ_X and θ_Y are the zero vectors of X and Y, respectively ;

(N2) $\phi(\cdot\,,I)$ is continuous in X;

(N3) $\phi(x,I_1\cup I_2) = \phi(x,I_1) + \phi(x,I_2)$ whenever I_1 and I_2 are disjoint and adjacent;

(N4) given $M > 0$, for every $\varepsilon > 0$ there exists an $\eta > 0$ such that

$$\left\| \sum_{i=1}^{n} \phi(x_i, I_i) - \sum_{i=1}^{n} \phi(y_i, I_i) \right\| < \varepsilon$$

whenever $\|x_i - y_i\| < \eta$, $\|x_i\| \le M$, $\|y_i\| \le M$, for $i = 1, 2, ..., n$ and the intervals $I_1, I_2, I_3, ..., I_n$ are pairwise disjoint;

(N5) given $M > 0$, for every $\varepsilon > 0$ there exists an $\eta > 0$ such that

$$\left\| \sum_{i=1}^{n} \phi(x_i, I_i) \right\| < \varepsilon$$

whenever $I_1, I_2, ..., I_n$ are pairwise disjoint intervals in $[a,b]$ with total length less that η and $\|x_i\| \le M$ for all $i = 1, 2, ..., n$.

Moreover, we shall assume that

$$\phi(x, [u,v]) = \phi(x, (u,v]) = \phi(x, [u,v)) = \phi(x, (u,v)).$$

Remark: Condition (N1) ensures that the ϕ-integral of the zero function is the zero vector θ_Y.

2. Results and Examples

We now give the simple properties of the integral.

Theorem 2. *Let $a < c < b$. If $f : [a,b] \to X$ is ϕ-integrable on $[a,c]$ and on $[c,b]$, then so it is on $[a,b]$ and*

$$\int_a^b \phi(f(t), dt) = \int_a^c \phi(f(t), dt) + \int_c^b \phi(f(t), dt).$$

Proof. Let A and B be the ϕ-integrals of f on $[a,c]$ and on $[c,b]$, respectively. Then, given $\varepsilon > 0$, there exists a $\delta_1(\xi) > 0$ on $[a,c]$ such that if $D_1 = \{([u,v];\xi)\}$ is δ_1-fine division of $[a,c]$, we have

$$\left\|(D_1)\sum \varphi(f(\xi),[u,v]) - A\right\| < \frac{\varepsilon}{2}.$$

Similarly, there exists a $\delta_2(\xi) > 0$ on $[a,c]$ such that if $D_2 = \{([u,v];\xi)\}$ is a δ_2-fine division of $[c,b]$, we have

$$\left\|(D_2)\sum \varphi(f(\xi),[u,v]) - B\right\| < \frac{\varepsilon}{2}.$$

Now define $\delta(\xi) > 0$ on $[a,b]$ as follows:

$$\delta(\xi) = \begin{cases} \min\{\delta_1(\xi),\ c-\xi\}, & \text{if } \xi \in [a,c), \\ \min\{\delta_1(\xi),\ \delta_2(\xi)\}, & \text{if } \xi = c, \\ \min\{\delta_2(\xi),\ \xi - c\}, & \text{if } \xi \in (c,b]. \end{cases}$$

Then for any δ-fine division of $[a,b]$, c is always an associated point.

Let $D = \{([u,v];\xi)\}$ be a δ-fine division of $[a,b]$. If c is a division point, then let D_1 be the division of $[a,c]$ consisting of intervals $[u,v]$ in D with $[u,v] \subset [a,c]$ and D_2 the division of $[c,b]$ consisting of intervals $[u,v]$ in D with $[u,v] \subset [c,b]$. Otherwise, i.e., if there is an interval $[u',v']$ in D with $c \in (u',v')$, then we let D_1 be the division consisting of the interval $[u',c]$ and all intervals $[u,v]$ in D with $[u,v] \subset [a,c)$ and D_2 the division of $[c,b]$ consisting of the interval $[c,v']$ and all intervals $[u,v]$ in D with $[u,v] \subset (c,b]$. Note that by (N3) we have

$$\phi(f(c),[u',v']) = \phi(f(c),[u',c]) + \phi(f(c),[c,v']).$$

Also, we find that D_1 is a δ_1-fine division of $[a,c]$ and D_2 is a δ_2-fine division of $[c,b]$. Thus

$$\left\|(D)\sum \phi(f(\xi),[u,v]) - (A+B)\right\| \leq \left\|(D_1)\sum \phi(f(\xi),[u,v]) - A\right\| +$$

$$+\left\|(D_2)\sum \phi(f(\xi),[u,v]) - B\right\| < \varepsilon.$$

Therefore f is ϕ-integrable on $[a,b]$ and

$$\int_a^b \phi(f(t), dt) = A + B. \quad \square$$

The proofs of the following two results are similar to those that are found in [7]. The first, which is known as the Cauchy criterion, is used to prove the second.

Theorem 3. *A function* $f:[a,b] \to X$ *is* ϕ*-integrable on* $[a,b]$ *if and only if for every* $\varepsilon > 0$, *there exists a* $\delta(\xi) > 0$ *defined on* $[a,b]$ *such that for any two* δ*-fine divisions* $D_1 = \{([u,v];\xi)\}$ *and* $D_2 = \{([u',v'];\xi')\}$ *of* $[a,b]$,

$$\left\|(D_1)\sum \phi(f(\xi),[u,v]) - (D_2)\sum \phi(f(\xi'),[u',v'])\right\| < \varepsilon.$$

Theorem 4. *If* $f:[a,b] \to X$ is ϕ*-integrable on* $[a,b]$, *then so it is on a subinterval* $[c,d]$ *of* $[a,b]$.

Definition 5. Let $f:[a,b] \to X$ be ϕ-integrable on $[a,b]$. The function $F_\phi : [a,b] \to Y$, defined by

$$F_\phi(t) = \int_a^t \phi(f(t'), dt'),$$

is called a ***ϕ-primitive*** of f on $[a,b]$. If $[u,v] \subset [a,b]$, then

$$F_\phi(u,v) = F_\phi(v) - F_\phi(u)$$

$$= \int_a^v \phi(f(t),dt) - \int_a^u \phi(f(t),dt) = \int_u^v \phi(f(t),dt).$$

Cao proved in [3] that if F is a primitive of a Henstock-Bochner integrable function $f:[a,b] \to X$, then F is continuous on $[a,b]$. Here, we give an analogous result for the ϕ-integral. First, we state without proof the following simple lemma.

Lemma 6. *Let $f: [a,b] \to X$ be ϕ-integrable on $[a,b]$ with ϕ-primitive F_ϕ. Then for every $\varepsilon > 0$ there exists a $\delta(\xi) > 0$ on $[a,b]$ such that if $D = \{([u,v];\xi)\}$ is a partial division of $[a,b]$, we have*

$$\left\| (D)\sum \left\{ \phi(f(\xi),[u,v] - F_\phi(u,v) \right\} \right\| < \varepsilon.$$

Theorem 7. *Let $f: [a,b] \to X$ be ϕ-integrable on $[a,b]$ with ϕ-primitive F_ϕ : $[a,b] \to Y$ given in Definition 5. Then F_ϕ is continuous on $[a,b]$.*

Proof. Let $\varepsilon > 0$. By Lemma 6, there exists a $\delta(\xi) > 0$ such that if $D' = \{([u,v];\xi)\}$ is a partial division of $[a,b]$, we have

$$\left\| (D')\sum \{ \phi(f(\xi),[u,v]) - F_\phi(u,v) \} \right\| < \varepsilon.$$

Let $\xi' \in [a,b]$ and let $M = \|f(\xi')\|$. Then in view of condition (N5), there exists $\eta > 0$ such that

$$\|\phi(f(\xi'),I)\| < \varepsilon$$

whenever the length of I is less than η. Define $\delta^*(\xi) = \min\{\delta(\xi), \eta\}$ for each $\xi \in [a,b]$. Then every δ^*-fine division is a δ-fine division.

Next, let $t \in [a,b]$ be such that $|t - \xi'| < \delta^*(\xi')$. Without loss of generality, we may assume that $\xi' < t$. Choose a δ^*-fine division D of $[a,b]$ in such a way that $[\xi',t]$ is an interval in D with ξ' as the associated point. Then

$$\left\| F_\phi(t) - F_\phi(\xi') - \phi(f(\xi'),[\xi',t]) \right\| < \varepsilon.$$

Also, since $\|f(\xi')\| = M$ and $|t - \xi'| < \eta$, it follows that

$$\|\phi(f(\xi'),[\xi',t])\| < \varepsilon.$$

Therefore,

$$\left\| F_\phi(t) - F_\phi(\xi') \right\| \le \left\| F_\phi(\xi',t) - \phi(f(\xi'),[\xi',t]) \right\| + \|\phi(f(\xi'),[\xi',t])\| < 2\varepsilon.$$

This shows that F_ϕ is continuous. □

Theorem 8. *Let $f, g : [a,b] \to X$ be ϕ-integrable functions with ϕ-primitives F_ϕ and G_ϕ, respectively. Then $f + g$ is ϕ-integrable on $[a,b]$ whenever $\|f(t)\|\|g(t)\| = 0$ for all $t \in [a,b]$.*

Proof. Assume that $\|f(t)\|\|g(t)\| = 0$ for all t in $[a,b]$. Let $\varepsilon > 0$. Since f and g are ϕ-integrable on $[a,b]$, there exists a common $\delta(\xi) > 0$ such that for any δ-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$\left\|(D)\sum \phi(f(\xi),[u,v]) - F_\phi(a,b)\right\| < \frac{\varepsilon}{2} \quad \text{and}$$

$$\left\|(D)\sum \phi(g(\xi),[u,v]) - G_\phi(a,b)\right\| < \frac{\varepsilon}{2}.$$

By condition (N1) and our assumption, we have

$$\phi(f(\xi) + g(\xi), [u,v]) = \phi(f(\xi),[u,v]) + \phi(g(\xi),[u,v]) \text{ for all } \xi \in [a,b].$$

Therefore,

$$\left\|(D)\sum \phi(f(\xi) + g(\xi),[u,v]) - [F_\phi(a,b) + G_\phi(a,b)]\right\|$$

$$\leq \left\|(D)\sum \phi(f(\xi),[u,v]) - F_\phi(a,b)\right\| + \left\|(D)\sum \phi(g(\xi),[u,v]) - G_\phi(a,b)\right\|$$

$$< \varepsilon.$$

This shows that $f + g$ is ϕ-integrable on $[a,b]$ and

$$\int_a^b \phi((f+g)(t),dt) = \int_a^b \phi(f(t),dt) + \int_a^b \phi(g(t),dt). \quad \square$$

The following are few examples of ϕ-integrable functions.

Proposition 9. *Let $f : [a,b] \to X$ be defined by $f(t) = \theta_X$ almost everywhere in $[a,b]$. Then f is ϕ-integrable on $[a,b]$ and*

$$\int_a^b \phi(f(t),dt) = \theta_Y.$$

Proof. Let $E = \{t \in [a,b] : \|f(t)\| \neq 0\}$. Then E is of measure zero. For each $i \geq 1$, let $E_i = \{t \in E : i - 1 < \|f(t)\| \leq i\}$. Then each E_i is of measure zero and E is the union of the E_i's.

Let $\varepsilon > 0$. Then by (N5), for each $i = 1, 2, \ldots$, there exists $\eta_i > 0$ such that

$$\left\| \sum_{j=1}^{n} \phi(x_j, I_j) \right\| < \varepsilon 2^{-i}$$

whenever the I_j's are pairwise disjoint intervals with total length less than η_i and $\|x_i\| \leq i$ for $j = 1, 2, \ldots, n$.

Since E_i is of measure zero, there exists an open set G_i such that $E_i \subset G_i$ and G_i is of measure less than η_i. Define $\delta(\xi) > 0$ so that $(\xi - \delta(\xi), \xi + \delta(\xi)) \subset G_i$ when $\xi \in E_i$, and arbitrarily if otherwise. For any δ-fine division D = $\{([u,v];\xi)\}$ of $[a,b]$, we have

$$\left\|(D)\sum \phi(f(\xi),[u,v])\right\| = \left\| \sum_{\xi \in E} \phi(f(\xi),[u,v]) \right\| + \left\| \sum_{\xi \notin E} \phi(f(\xi),[u,v]) \right\|$$

$$= \left\| \sum_{\xi \in E} \phi(f(\xi),[u,v]) \right\|$$

$$\leq \sum_i \left\| \sum_{\xi \in E_i} \phi(f(\xi),[u,v]) \right\|$$

$$\leq \sum_i \varepsilon 2^{-i} < \varepsilon .$$

Therefore, f is ϕ-integrable on $[a,b]$ and

$$\int_a^b \phi(f(t), dt) = \theta_Y . \square$$

Proposition 10. *Let I be a subinterval of $[a,b]$ and $x \in X$. Consider the function $f : [a,b] \to X$ defined by $f(t) = x\chi_I(t)$ for every t in $[a,b]$, where χ_I is the characteristic function of I. Then f is ϕ-integrable on $[a,b]$ and*

$$\int_a^b \phi(f(t), dt) = \phi(x, I) .$$

Proof. Let t_1 and t_2 $(t_1 < t_2)$ be the endpoints of the interval I and $\varepsilon > 0$. Define $\delta(\xi) > 0$ such that

(i) $(\xi - \delta(\xi), \xi + \delta(\xi)) \subset \mathrm{int}(I)$, if $\xi \in \mathrm{int}(I)$

(*ii*) $\delta(\xi) \le \dfrac{t_2 - t_1}{2}$, if $\xi = t_1$ or $\xi = t_2$.

(*iii*) $\delta(\xi) < t_1 - \xi$, if $\xi < t_1$ and

(*iv*) $\delta(\xi) < \xi - t_2$, if $\xi > t_2$.

Further, we choose $\delta(\xi)$ so that $\delta(t_1)$ and $\delta(t_2)$ are both less than $\eta/8$, where η is chosen as in (N5) (corresponding to $\varepsilon > 0$ and $M = \|x\|$).

Let D = $\{([u,v];\xi)\}$ be a δ-fine division of $[a,b]$. Denote by D* the partial division consisting of all intervals $[u,v]$ in D with $[u,v] \subset \text{int}(I)$. Let $[u',v']$ and $[u'',v'']$ be the intervals in D such that $v', u'' \in \text{int}(I)$ and $t_1 \in [u',v']$ and $t_2 \in [u'',v'']$. Then, by (N3),

$$\phi(x,I) = (\text{D*})\sum \phi(x,[u,v]) + \phi(x,[t_1,v']) + \phi(x,[u'',t_2]).$$

Thus,

$$\left\|(\text{D})\sum \phi(f(\xi),[u,v]) - (\text{D*})\sum \phi(x,[u,v]) - \phi(x,[t_1,v']) - \phi(x,[u'',t_2])\right\|$$

$$= \left\|(\text{D}) \sum_{\xi \in \bar{I}} \phi(f(\xi),[u,v]) - (\text{D*})\sum \phi(x,[u,v]) - \phi(x,[t_1,v']) - \phi(x,[u'',t_2])\right\|$$

$$= \left\| \sum_{\xi = t_1,t_2} \phi(f(\xi),[u,v]) - \phi(x,[t_1,v']) - \phi(x,[u'',t_2])\right\|$$

$$= \left\| \sum_{\xi = t_1,t_2} \phi(f(\xi),[u,v])\right\| + \|\phi(x,[t_1,v'])\| + \|\phi(x,[u'',t_2])\|$$

$< 3\varepsilon$.

Therefore, f is ϕ-integrable on $[a,b]$ and

$$\int_a^b \phi(f(t),dt) = \int_{t_1}^{t_2} \phi(f(t),dt) = \phi(x,I). \ \square$$

We remark that in Example 10, f is called a single step function. Also, whether I is $[t_1,t_2]$ or $[t_1,t_2)$ or $(t_1,t_2]$ or (t_1,t_2), the same integral of f is obtained. This means that changing one point does not change the value of the ϕ-integral.

Proposition 11. *Let $f: [a,b] \to X$ be a step function, or more precisely, let $\{I_1, I_2, ..., I_n\}$ be a division of $[a,b]$, $x_1, x_2, ..., x_n$ be elements of X and $f(t) = x_i$ when $t \in I_i$. Then f is ϕ-integrable on $[a,b]$ and*

$$\int_a^b \phi(f(t), dt) = \sum_{i=1}^{n} \phi(x_i, I_i).$$

Proof. Let t_{i-1} and t_i $(t_{i-1} < t_i)$ be the endpoints of I_i. Put $f_i(t) = x_i$ when $t \in I_i$ and zero elsewhere. Then by Example 10, each f_i is ϕ-integrable on $[a,b]$ and

$$\int_a^b \phi(f_i(t), dt) = \int_{t_{i-1}}^{t_i} \phi(f_i(t), dt) = \phi(x_i, I_i).$$

By Theorem 2, f is ϕ-integrable on $[a,b]$ and

$$\int_a^b \phi(f(t), dt) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \phi(f_i(t), dt)$$

$$= \sum_{i=1}^{n} \phi(x_i, I_i). \quad \square$$

Propostion 12. *Let $f: [a,b] \to X$ be a continuous function. Then f is ϕ-integrable on $[a,b]$.*

Proof. The key is to find a suitable $\delta(\xi) > 0$ when given $\varepsilon > 0$. To this end, let $\varepsilon > 0$ and put $M = \sup \{\|f(t)\| : a \le t \le b\}$. Then by (N4) there exists $\eta > 0$ such that

$$\left\| \sum_{i=1}^{n} \phi(x_i, I_i) - \sum_{i=1}^{n} \phi(y_i, I_i) \right\| < \varepsilon$$

whenever $\|x_i - y_i\| < \eta$, $\|x_i\| \le M$, $\|y_i\| \le M$ for $i = 1, 2, ..., n$ and $I_1, I_2, I_3, ..., I_n$ are pairwise disjoint.

Since f is continuous on $[a,b]$, it is uniformly continuous there. Hence there is $\delta > 0$ such that $\|f(t) - f(t')\| < \eta$ whenever $|t - t'| < \delta$. Define $\delta^*(\xi) = \delta/2$ for all ξ in $[a,b]$. For any division $D = \{([u,v];\xi)\}$, write

$$\sigma(f, D) = (D)\sum \phi(f(\xi), [u,v]).$$

Then if D_1 and D_2 are any two δ^*-fine divisions of $[a,b]$, there exists a δ^*-fine division D_3 of $[a,b]$ which is finer than both D_1 and D_2. Thus

$$\|\sigma(f,D_1)-\sigma(f,D_2)\| \le \|\sigma(f,D_1)-\sigma(f,D_3)\| + \|\sigma(f,D_3)-\sigma(f,D_2)\|$$

$$< 2\varepsilon .$$

By the Cauchy criterion, it follows that f is ϕ-integrable on $[a,b]$. □

Finally, we state the following convergence theorems.

Theorem 13. *Let $\{f_n : [a,b] \to X\}$ be a sequence of ϕ-integrable functions on $[a,b]$. If $\{f_n\}$ converges uniformly to a bounded function $f: [a,b] \to X$, then f is ϕ-integrable and*

$$\int_a^b \phi(f(t),dt) = \lim_{n\to\infty} \int_a^b \phi(f_n(t),dt).$$

Proof. First, we show that $\lim_{n\to\infty} \int_a^b \phi(f_n(t),dt)$ exists. So, let $\varepsilon > 0$. Then there exists a natural number N_1 such that for all $n \ge N_1$ and for all t in $[a,b]$, we have

$$\|f_n(t) - f(t)\| < \varepsilon.$$

It then follows that there exists a positive real number M such that for all $n \ge N_1$ and for all t in $[a,b]$, $\|f_n(t)\| \le M$. In view of condition (N4), given $M > 0$ and $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\left\|\sum_{i=1}^{m} \phi(x_i,I_i) - \sum_{i=1}^{m} \phi(y_i,I_i)\right\| < \varepsilon$$

whenever $\|x_i - y_i\| < \eta$, $\|x_i\| \le M$, and $\|y_i\| \le M$ for $i = 1, 2, ..., m$ and $I_1, I_2, ..., I_m$ are non-overlapping intervals in $[a,b]$.

For $\eta > 0$, there exists a natural number N_2 such that for all $n \ge N_2$ and for all t in $[a,b]$, we have

$$\|f_n(t) - f(t)\| < \frac{\eta}{2}.$$

Choose $N = \max\{N_1, N_2\}$. Then for all $n, m \geq N$ and for all t in $[a,b]$, we find that $\|f_n(t)\| \leq M$, $\|f_m(t)\| \leq M$, and $\|f_n(t) - f_m(t)\| < \eta$.

Now let $n, m \geq N$. Since f_n and f_m are ϕ-integrable, there exists a $\delta(\xi) > 0$ such that for any δ-fine division D = $\{([u,v];\xi)\}$ of $[a,b]$, we have

$$\left\|(D)\sum \phi(f_n(\xi),[u,v]) - \int_a^b \phi(f_n(t),dt)\right\| < \varepsilon \text{ and}$$

$$\left\|(D)\sum \phi(f_m\xi),[u,v]) - \int_a^b \phi(f_m(t),dt)\right\| < \varepsilon.$$

Thus

$$\left\|\int_a^b \phi(f_n(t),dt) - \int_a^b \phi(f_m(t),dt)\right\| \leq \left\|\int_a^b \phi(f_n(t),dt) - (D)\sum \phi(f_n(\xi),[u,v])\right\| +$$

$$+\left\|(D)\sum \phi(f_n(\xi),[u,v]) - (D)\sum \phi(f_m(t),[u,v])\right\| +$$

$$+\left\|(D)\sum \phi(f_m(\xi),[u,v]) - \int_a^b \phi(f_m(t),dt)\right\|$$

$$< 3\varepsilon.$$

This shows that the sequence $\{\int_a^b \phi(f_n(t),dt)\}$ is a Cauchy sequence in Y. Since Y is complete, there exists $A \in Y$ such that

$$A = \lim_{n\to\infty} \int_a^b \phi(f_n(t),dt).$$

It remains to show that A is the ϕ-integral of f on $[a,b]$. To this end, let $\varepsilon > 0$. Then there exist a natural number N and a positive number M such that

$$\|f_N(t) - f(t)\| < \eta,$$

$$\left\|\int_a^b \phi(f_N(t),dt) - A\right\| < \varepsilon,$$

and $\|f_N(t)\| \leq M$, $\|f(t)\| \leq M$ for all $t \in [a,b]$, where η is chosen as in (N4). Since f_N is ϕ-integrable on $[a,b]$, there exists a $\delta_N(\xi) > 0$ such that for any δ_N-fine division D = $\{([u,v];\xi)\}$ of $[a,b]$, we have

$$\left\| \int_a^b \phi(f_N(t), dt) - (D)\sum \phi(f_N(\xi), [u,v]) \right\| < \varepsilon.$$

Define $\delta(\xi) = \delta_M(\xi)$ for all ξ in $[a,b]$. Then for any δ-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$\begin{aligned} \left\| (D)\sum \phi(f(\xi),[u,v]) - A \right\| &\le \left\| (D)\sum \phi(f(\xi),[u,v]) - (D)\sum \phi(f_N(\xi),[u,v]) \right\| + \\ &+ \left\| (D)\sum \phi(f_N(\xi),[u,v]) - \int_a^b \phi(f_N(t),dt) \right\| + \\ &+ \left\| \int_a^b \phi(f_n(t),dt) - A \right\| \\ &< 3\varepsilon . \end{aligned}$$

Therefore, f is ϕ-integrable on $[a,b]$ and

$$\int_a^b \phi(f(t),dt) = \lim_{n\to\infty} \int_a^b \phi(f_n(t),dt). \quad \square$$

Corollary 14. *If* $\{f_n : [a,b] \to X\}$ *is a sequence of* ϕ*-integrable bounded functions on* $[a,b]$ *which converges uniformly to* $f : [a,b] \to X$*, then* f *is* ϕ*-integrable on* $[a,b]$ *and*

$$\int_a^b \phi(f(t),dt) = \lim_{n\to\infty} \int_a^b \phi(f_n(t),dt).$$

Proof. It follows from the hypothesis that f is bounded. Thus, in view of Theorem 13, we have the desired result. $\square$

Acknowledgment. The author would like to thank the referees for their worthwhile suggestions and comments on the earlier draft of this paper.

References

[1] Canoy, S., *Nonlinear Henstock-Kurzweil Integral in Banach Spaces,* Ph.D. Thesis, U.P. Diliman (1994)

[2] Cao, S., *The Henstock integral for Banach-valued functions,* SEA Bull. Math **16** (2) (1992), 35-40

[3] Cao, S., *Henstock Integration in Banach Spaces*, Ph.D. Thesis, U.P. Diliman (1990)

[4] Chew, T. S., *Nonlinear Henstock-Kurzweil integral and representation theorems*, SEA Bull. Math **12** (1988), 97-108

[5] Dinculeanu, N., *Contributions of Romanian mathematicians to the measure and integration theory*, Rev. Roumaine Math Pures **11** (1996), 1075-1102

[6] Drewnowski L., and Orlicz, W., *Continuity and representation theorems of orthogonally additive functionals*, Bull. Acad. Polon. Sci. Ser. Sci. Math, Astron. et Phys. **17** (1969), 647-653

[7] Lee, P.Y., *Lanzhou Lectures on Henstock Integration*, World Scientific, (1989)