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The Span of Graphs Resulting from Deletion of Edges

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Abstract

Let S be a set of edges in the complete graph K_n. This paper gives the span of $K_n - S$ in some Euclidean spaces. Among others, it considers the cases when S is a singleton and when S is independent.

1. Introduction

By a graph we mean a simple graph. One graph that is of importance in this study is the complete graph K_n of order n. The reader may refer to [H] for other terms and concepts whose definitions are not given here.

We shall refer to $\mathbf{R}^n$ as the Euclidean n-space with the Euclidean distance. For convenience, the Euclidean 0-space will be understood to contain only the zero point. The ***centroid*** of m points $(x_{k1}, x_{k2}, \ldots, x_{kn}) \in \mathbf{R}^n$ is the point $(z_1, z_2, \ldots, z_n) \in \mathbf{R}^n$, where

$$z_s = \frac{1}{m}(x_{1s} + x_{2s} + \cdots + x_{ms}).$$

Definition 1. A graph G is a ***unit graph*** in $\mathbf{R}^n$ if there is a one-to-one mapping $\phi : V(G) \to \mathbf{R}^n$ such that the following conditions are satisfied:

1. If $[x,y] \in E(G)$, then $|\phi(x) - \phi(y)| = 1$ and the line segment $[\phi(x),\phi(y)]$ joining $\phi(x)$ and $\phi(y)$ is an edge of G in $\mathbf{R}^n$.

2. If $[x,y]$ and $[u,v]$ have no common vertex, then the line segments $[\phi(x),\phi(y)]$ and $[\phi(u),\phi(v)]$ have at most one point in common.

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Definition 2. The ***dimension*** of a graph G, denoted by $dim(G)$, is the smallest integer n such that G is a unit graph in $\mathbf{R}^n$.

Theorem 3. [G1] $dim(K_n) = n - 1$.

Definition 4. Let G be a unit graph in $\mathbf{R}^n$. The ***span*** of G in $\mathbf{R}^n$, denoted by $span_n(G)$, is the real number s such that for every $\varepsilon > 0$, the following conditions are satisfied:

1. There exists a unit representation of G in $\mathbf{R}^n$ which is contained in some open ball of diameter $s + \varepsilon$.
2. No unit representation of G in $\mathbf{R}^n$ is contained in any open ball of diameter s.

Theorem 3 tells us that the span of the complete graph K_n is defined only in Euclidean d-space where $d \geq n - 1$.

Theorem 5. [U] $span_d(K_n) = \sqrt{\frac{2(n-1)}{n}}$ *for* $d \geq n - 1$.

Definition 6. Let G be a graph whose vertices are points in $\mathbf{R}^n$ and whose edges are line segments connecting two vertices. Then G is ***flexible*** if its vertices can be continuously moved in $\mathbf{R}^n$ while preserving the length of all edges so that at least two vertices change their mutual distance. If G is not flexible in $\mathbf{R}^n$, it is said to be ***rigid*** in $\mathbf{R}^n$ or ***n-rigid***.

Remark 7. K_n is d-rigid for $d \geq n - 1$.

The next theorem introduces an operation which, when applied to a d-rigid graph, produces a 'bigger' d-rigid graph.

Theorem 8. [T] *Let G be a graph. Choose $d \geq 1$ distinct vertices $v_1, v_2, \ldots, v_d$ of G and add a new vertex v_0 together with d edges $v_0v_1, v_0v_2, \ldots, v_0v_d$ to G. Denote the resulting graph by A_dG. Then G is d-rigid if and only if A_dG is d-rigid.*

2. Deletion of Edges

Let G be a graph. If e is an edge of G, then $G - e$ denotes the maximal spanning subgraph of G not containing e. If S is a subset of the vertex-set of G, then $G - S$ denotes the subgraph of G resulting from the removal of single elements from S in succession.

Theorem 9. [G2] *Let e be an edge of the complete graph K_n of order $n \geq 3$. Then $dim(K_n - e) = n - 2$.*

Let e be an edge of the complete graph K_n of order $n \geq 3$. It follows from Theorem 9 that the span of $K_n - e$ in $\mathbf{R}^m$ is defined only for each $m \geq n - 2$.

Theorem 10. *Let e be an edge of the complete graph K_n of order $n \geq 3$. Then*

$$span_{n-2}(K_n - e) = \sqrt{\frac{2(n-1)}{n-2}}.$$

Proof. Let $V(K_n) = \{ 1, 2, \ldots, n \}$ and $e = [n,n-1]$. Then the vertices $1, 2, \ldots, n-1$ induce a complete subgraph K_{n-1} of $K_n - e$. Observe that $K_n - e$ can be obtained from K_{n-1} by adding the vertex n together with the edges $[1,n]$, $[2,n]$, $\ldots$, $[n-2,n]$. Now K_{n-1} is $(n-2)$-rigid by Remark 7. By Theorem 8, $K_n - e$ is $(n-2)$-rigid.

Consider the complete subgraph K_{n-2} of $K_n - e$ with $\{1, 2, \ldots, n-2\}$ as its vertex-set. Associate the vertex i with the point

$$p_i = (0, \ldots, 0, \sqrt{2}/2, 0, \ldots, 0) \in \mathbf{R}^{n-2},$$

where the coordinates are all 0 except the ith which is $\sqrt{2}/2$. Then $|p_i - p_j| = 1$ for any two distinct indices i and j. Let $p_{n-1} = (\alpha, \alpha, \ldots, \alpha)$ and $p_n = (\beta, \beta, \ldots, \beta)$ be points in $\mathbf{R}^{n-2}$, where

$$\alpha = \frac{\sqrt{2}(1+\sqrt{n-1})}{2(n-2)} \quad \text{and} \quad \beta = \frac{\sqrt{2}(1-\sqrt{n-1})}{2(n-2)}.$$

Then $|p_{n-1} - p_i| = |p_n - p_i| = 1$ for $i = 1, 2, \ldots, n-2$. Respectively, associate now the vertices $n-1$ and n with the points p_{n-1} and p_n. Thus, a unit representation of $K_n - e$ in $\mathbf{R}^{n-2}$ is obtained.

Let $c = (z_1, z_2, \ldots, z_{n-2})$ be the centroid of the points $p_1, p_2, \ldots, p_n$. Then

$$z_i = \frac{1}{n}\left(\frac{\sqrt{2}}{2} + \alpha + \beta\right) = \frac{\sqrt{2}}{2(n-2)}$$

for each i. Now

$$|p_i - c|^2 = \left[\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2(n-2)}\right]^2 + (n-3)\left[0 - \frac{\sqrt{2}}{2(n-2)}\right]^2 = \frac{n-3}{2(n-2)}$$

for $i = 1, 2, \ldots, n-2$ and

$$|p_{n-1} - c|^2 = (n-2)\left[\frac{\sqrt{2}(1+\sqrt{n-1})}{2(n-2)} - \frac{\sqrt{2}}{2(n-2)}\right]^2 = \frac{n-1}{2(n-2)},$$

$$|p_n - c|^2 = (n-2)\left[\frac{\sqrt{2}(1-\sqrt{n-1})}{2(n-2)} - \frac{\sqrt{2}}{2(n-2)}\right]^2 = \frac{n-1}{2(n-2)}.$$

Thus a ball of diameter $\sqrt{2(n-1)/(n-2)}$ inscribes the points $p_1, p_2, \ldots, p_n$. The conclusion of the theorem now follows since $K_n - e$ is $(n-2)$-rigid. □

Corollary 11. *Let S be a nonempty set of edges in the complete graph K_n of order $n \geq 3$. Then*

$$span_{n-2}(K_n - S) \leq \sqrt{\frac{2(n-1)}{n-2}}.$$

Theorem 12. *Let e be an edge of the complete graph K_n of order $n \geq 2$. Then*

$$span_m(K_n - e) = \sqrt{\frac{2(n-1)}{n-2}}$$

for $m \geq n$.

Proof. The case $n = 2$ is immediate. Suppose now that $n \geq 3$. It is sufficient to show the case $m = n$. For each integer k, $1 \leq k \leq n-1$, let

$$p_k = (0, \ldots, 0, \sqrt{2}/2, 0, \ldots, 0) \in \mathbf{R}^n,$$

where the coordinates are all 0 except the kth which is $\sqrt{2}/2$. Then the points p_k induce a unit K_{n-1} in $\mathbf{R}^n$. Note that

$$p_{n-1} \in C = \{(0, \ldots, 0, a, b) \in \mathbf{R}^n : a^2 + b^2 = 1/2\}.$$

Let $q \in C$ which is arbitrarily close to p_{n-1}. Now, the points p_k and q induce a unit $K_n - e$ in $\mathbf{R}^n$. By Theorem 5,

$$span_n(K_n - e) \le span_n(K_{n-1}) = \sqrt{2(n-2)/(n-1)}.$$

This inequality is actually an equality since $K_n - e$ contains a complete subgraph K_{n-1}. □

Corollary 13. *Let S be a nonempty set of edges in the complete graph K_n of order $n \ge 2$. Then*

$$span_m(K_n - S) \le \sqrt{\frac{2(n-2)}{n-1}}.$$

for $m \ge n$.

The next theorem cites some cases where Theorem 12 can be improved.

Theorem 14. *Let e be an edge of the complete graph K_n of order n. If $3 \le n \le 5$, then*

$$span_m(K_n - e) = \sqrt{\frac{2(n-2)}{n-1}}$$

for $m \ge n - 1$.

Proof. Let $n = 3$. Also, let $p_1 = (0,0)$ and $p_2 = (0,0)$. Take p_3 to be a point in $C = \{ (a, b) : a^2 + b^2 = 1 \}$ which is arbitrarily close to p_2. Then the points p_1, p_2, and p_3 form a unit $K_3 - e$ in $\mathbf{R}^2$.

Let $n = 4$. Also, let $p_1 = (0,1/2,0)$, $p_2 = (0,-1/2,0)$ and $p_3 = (\sqrt{3}/2,0,0)$. The points p_1, p_2 induce a unit K_2 in $\mathbf{R}^3$. Let $C = \{ (a, 0, b) : a^2 + b^2 = 3/4 \}$. Then $p_3 \in C$ and $|q - p_i| = 1$ for $i = 1, 2$. Take a point $p_4 \in C$ which is arbitrarily close to p_3. Then the points p_1, p_2, p_3 and p_4 form a unit $K_4 - e$ in $\mathbf{R}^3$.

Let $n = 5$. Also, let $p_1 = (1/2,-\sqrt{3}/6,0,0)$, $p_2 = (-1/2,-\sqrt{3}/6,0,0)$, $p_3 = (0,\sqrt{3}/2,0,0)$, and $p_4 = (0,0,\sqrt{6}/3,0)$. The points p_1, p_2, p_3 induce a unit K_3 in $\mathbf{R}^4$. Let $C = \{ (0, 0, a, b) : a^2 + b^2 = 2/3\}$. Then $p_4 \in C$ and $|q - p_i| = 1$ for $i = 1, 2, 3$. Take a point $p_5 \in C$ which is arbitrarily close to p_4. Then the points p_1, p_2, p_3, p_4 and p_5 form a unit $K_5 - e$ in $\mathbf{R}^4$.

The conclusion in each of the cases above follows from Theorem 5. □

3. Deletion of Independent Edges

Definition 15. A set S of edges in a graph G is ***independent*** if the elements of S are mutually nonadjacent. The largest number of edges in such a set is called the ***independence number*** of G and is denoted by $\beta(G)$ or β.

Lemma 16. [U] $\beta(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Let S be an independent set of edges in K_n. The next theorem tells us in what Euclidean space is the span of $K_n - S$ defined.

Lemma 17. [I] *Let S be an independent set of edges in the complete graph* K_n *of order* $n \geq 4$. *If* $2 \leq |S| \leq \lfloor n/2 \rfloor$, *then* $dim(K_n - S) = n - |S|$.

From hereon we denote by nG the sum of n copies of graph G. By O_n we mean the complete n-partite graph $K(2, 2, \ldots, 2)$.

Lemma 18. [U] *For* $n \geq 3$,

1. O_n *is n-rigid*;
2. $span_n(O_n) = \sqrt{2}$.

Theorem 19. *Let S be an independent set of edges in the complete graph* K_n *with* $|S| = \beta = \lfloor n/2 \rfloor$. *Then*

(*a*) $K_n - S$ *is* β*-rigid for* $n \geq 6$;

(*b*) $span_\beta(K_n - S) = \sqrt{2}$ *for even* $n \geq 6$.

Proof. Let n be odd. Then $K_n - S = K_1 + \beta\overline{K_2} = K_1 + O_\beta$. Since $n \geq 7$, it follows from Lemma 18 that O_β is β-rigid. Now O_β has 2β vertices. We add another vertex to O_β and join it with its 2β vertices. The resulting graph is isomoporphic to $K_n - S$. It is β-rigid by Theorem 8.

Let n be even. Then $K_n - S = \beta\overline{K_2} = O_\beta$. Since $n \geq 6$, it follows from Lemma 18 that O_β is β-rigid and that $span_\beta(K_n - S) = \sqrt{2}$. □

Let S be an independent set of edges in the complete graph K_n with $|S| = \lfloor n/2 \rfloor$. The next theorem considers the span of $K_n - S$ for odd n. It provides an upper bound and a lower bound for the span of $K_n - S$ in lower dimensions (see also Corollary 11).

Theorem 20. *Let S be an independent set of the edges in the complete graph* K_n. *If* $|S| = k \geq 2$, *then*

$$\sqrt{\frac{2(n-k-1)}{n-k}} \leq span_{n-k}(K_n - S) \leq \sqrt{2}.$$

Proof. Let $\alpha = \sqrt{2}/2$. For $1 \leq i \leq n-k$, let $p_i = (0, \ldots, 0, \alpha, 0, \ldots, 0) \in \mathbf{R}^{n-k}$, where the coordinates are all 0 except the ith which is α. For $n-2k+1 \leq j \leq n-k$, let $q_i = (0, \ldots, 0, -\alpha, 0, \ldots, 0)$, where the coordinates are all 0 except the jth which is $-\alpha$. Then

$$S = \{[p_i, q_i] : i = n-2k+1, n-2k+2, \ldots, n-k\}$$

is a set of k independent edges. It follows that the points p_i and q_i form a unit representation of $K_n - S$ in $\mathbf{R}^{n-k}$. Thus $span_{n-k}(K_n - S) \leq \sqrt{2}$. But $K_n - S = k\overline{K_2} + K_{n-2k}$; hence, $K_n - S$ contains a complete subgraph K_{n-k}. In the unit representation of $K_n - S$ above, the points $p_1, p_2, \ldots, p_{n-k}$ actually form a unit representation of K_{n-k} in $\mathbf{R}^{n-k}$ Since K_{n-k} is rigid in $\mathbf{R}^m$ for each $m \geq n-k-1$, we have

$$span_{n-k}(K_{n-k}) = span_{n-k-1}(K_{n-k}) = \sqrt{\frac{2(n-k-1)}{n-k}}$$

by Theorem 5. Thus, $span_{n-k}(K_{n-k}) \leq span_{n-k}(K_n - S)$ since K_{n-k} is a subgraph of $K_n - S$. □

An upper bound for the span of $K_n - S$ in Theorem 20 could be found in terms of n. Consider the unit representation of $K_n - S$ in $\mathbf{R}^{n-k}$, which is given in the proof of Theorem 20. Let $z = (z_1, z_2, \ldots, z_{n-k})$ be the centroid of the points p_i and q_j. Then

$$z_1 = z_2 = \cdots = z_{n-2k} = \alpha/n$$

and

$$z_{n-2k+1} = z_{n-2k+2} = \cdots = z_{n-k} = 0.$$

Now,

$$|p_i - z|^2 = (\alpha - \alpha/n)^2 + (n-2k-1)(\alpha/n)^2 = (\alpha/n)^2(n^2 - n - 2k)$$

for $1 \leq i \leq n-2k$, and

$$|p_i - z|^2 = |q_i - z|^2 = (n-2k)(\alpha/n)^2 + \alpha^2 = (\alpha/n)^2(n^2 + n - 2k)$$

for $n-2k+1 \leq i \leq n-k$. Thus,

$$span_{n-k}(K_n - S) \le \frac{2\alpha}{n}\sqrt{n^2 + n - 2k} = \frac{1}{n}\sqrt{2(n^2 + n - 2k)}.$$

Next, we have to compare this upper bound with $\sqrt{2}$. Since $k \le n/2$, it follows that

$$\frac{1}{n}\sqrt{2(n^2 + n - 2k)} \ge \frac{1}{n}\sqrt{2[n^2 + n - 2(n/2)]} = \sqrt{2}.$$

Therefore, the upper bound $\sqrt{2}$ in Theorem 20 is better than $\frac{1}{n}\sqrt{2(n^2 + n - 2k)}$.

Corollary 21. *Let S be an independent set of the edges in the complete graph K_n with $|S| = \beta(K_n) = \beta$. If n is odd, then*

$$\sqrt{\frac{2(n-1)}{n+1}} \le span_{\beta}(K_n - S) \le \sqrt{2}.$$

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