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The HSL Integral

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Abstract

In this paper, we introduce the HSL *integral. We give the elementary properties and state necessary and sufficient conditions for its existence. We also state and prove the absolute integrability theorem, and find conditions for equivalence with the Henstock-Stieltjes integral.*

1. Introduction

In this paper, we always write $\lambda\cdot v = v\cdot\lambda$ for all vectors v and scalars λ.

Definition 1.1. Let X *be* a (real) Banach space. Given any functions $g : [a,b] \to X$ (resp. $\mathbf{R}$) and $f: [a,b] \to \mathbf{R}$ (resp. X), we say that f is ***Henstock-Stieltjes integrable*** (or HS-integrable) with respect to g on $[a,b]$ to a vector $J \in X$, and write

$$J = (HS)\int_a^b f\,\mathrm{d}g,$$

if and only if for every $\varepsilon > 0$ there exists a positive function $\delta(\xi)$ on $[a,b]$ such that for all δ-fine divisions $D = \{([u,v];\xi)\}$ of $[a,b]$,

$$\left\|(D)\sum f(\xi)\{g(v) - g(u)\} - J\right\| < \varepsilon.$$

Recall that a division $D = \{([u,v];\xi)\}$ is δ-***fine*** if $\xi \in [u,v] \subset (\xi-\delta(\xi), \xi+\delta(\xi))$ for every interval-point pair $([u,v];\xi)$ in D. And any subset of D is called a δ-***fine partial division*** of $[a,b]$. As always, we put $(HS)\int_a^a f\,\mathrm{d}g = 0$.

A chapter of the book of McLeod [6] is focused on the integral in Definition 1.1 but in finite dimensional Banach spaces X. In [3], there is an attempt to investigate the properties of the integral in an arbitrary Banach space. A

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complete list of the common properties is given such as the Cauchy criterion for integrals, linearity over integrands and integrators, and linearity over subintervals of $[a,b]$. It is known that if the functions f and g in Definition 1.1 are regulated, then the existence of the integral is always guaranteed.

Now, let X be a (real) Banach space, and let $f: [a,b] \to \mathbf{R}$ (resp. X) be HS-integrable with respect to $g : [a,b] \to X$ (resp. $\mathbf{R}$). Consider the X-valued function F on $[a,b]$ given by

$$F(x) = (HS)\int_a^x f\,\mathrm{d}g$$

for all $x \in [a,b]$. For any partition $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$(D)\Sigma\{F(v) - F(u)\} = (HS)\int_a^b f\,\mathrm{d}g.$$

Lemma 1.2. *Let X be a Banach space. If $f: [a,b] \to \mathbf{R}$ (resp. X) is HS-integrable with respect to $g : [a,b] \to X$ (resp. $\mathbf{R}$), then there exists a function $F : [a,b] \to X$ with the property that given $\varepsilon > 0$, there is a positive function $\delta(\xi)$ on $[a,b]$ such that for all δ-fine divisions $D = \{([u,v];\xi)\}$ of $[a,b]$,*

$$\left\|(D)\sum\left\{F(v) - F(u) - f(\xi)\{g(v) - g(u)\}\right\}\right\| < \varepsilon.$$

The function F in Lemma 1.2 is called the ***primitive of f with respect to g on*** $[a,b]$. We shall adopt the notation $F(u,v)$ for $F(v) - F(u)$. Then we have

$$F(a,b) = (HS)\int_a^b f\,\mathrm{d}g.$$

We remark that two primitives differ by a constant vector.

For real-valued integrals, Henstock's lemma plays a significant role in the proofs of many important results in Henstock integration theory. This lemma is given in two equivalent versions. In the Stieltjes sense, the strong version of the lemma can be stated as follows: *Let f and g be real-valued functions on $[a,b]$. If f is* HS-*integrable with respect to g on $[a,b]$ with primitive F, then given $\varepsilon > 0$, there exists a positive function $\delta(\xi)$ on $[a,b]$ such that for all δ-fine divisions $D = \{([u,v];\xi)\}$ of $[a,b]$,*

$$(D)\Sigma|F(v) - F(u) - f(\xi)\{g(v) - g(v)\}| < \varepsilon.$$

Using the identity real-valued function g, it is shown in [2] that in an infinite dimensional space, a Henstock-Stieltjes integral need not satisfy the strong version of Henstock's lemma. That is, the above lemma cannot be proved when $|\cdot|$ is replaced by $\|\cdot\|$. The concern of this present paper is on Stieltjes integrals that have the property of Henstock's lemma. We will give the definition of the HSL integral, investigate its elementary properties and present a necessary and sufficient condition for existence. We will prove the absolute integrability property and show the relationship of the HSL integral with the integral in Definition 1.1.

The following theorem is found in [3]. We will need this result later.

Theorem 1.3. *Let X be a Banach space and X' the space of all bounded linear functionals on X. Let $f : [a,b] \to X$ (resp. $\mathbf{R}$) be HS-integrable with respect to $g : [a,b] \to \mathbf{R}$ (resp. X) on $[a,b]$ with primitive F. If $\varepsilon > 0$, then there exists $\delta(\xi) > 0$ on $[a,b]$ such that given a δ-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$, we have*

$$\sup_{\substack{T \in X' \\ \|T\| \le 1}} (D)\sum \left| T\big(F(v) - F(u) - f(\xi)\{g(v) - g(u)\}\big)\right| < \varepsilon.$$

2. Results

Throughout the remaining discussion, X and Y denote real Banach spaces.

Definition 2.1. For any functions $f : [a,b] \to X$ (resp. $\mathbf{R}$) and $g : [a,b] \to \mathbf{R}$ (resp. X), we say that f is ***HSL-integrable*** with respect to g on $[a,b]$ whenever there exists a function $F : [a,b] \to X$ with the property: Given $\varepsilon > 0$, there is a positive function $\delta(\xi)$ on $[a,b]$ such that for all δ-fine divisions $D = \{([u,v];\xi)\}$ of the interval $[a,b]$,

$$(D)\Sigma \,|\, F(u,v) - f(\xi)\{g(v) - g(u)\} \,|\, < \varepsilon.$$

Such an integral is called an **HSL *integral***. When it exists, we call F a ***primitive*** of f with respect to g on $[a,b]$ in the HSL sense. If no confusion arises, we simply omit "in the HSL sense", and we write

$$F(a,b) = (HSL)\int_a^b f\,dg\,.$$

Once the inequality in Definition 2.1 holds, it also holds when D is replaced by a δ-fine partial division of $[a,b]$. By the triangle inequality, if f is HSL-integrable with respect to g on $[a,b]$, then f must be HS-integrable with respect to g on $[a,b]$ and the two integrals coincide. Moreover, in this case, a primitive of f with respect to g in the HSL sense is a primitive of f with respect to g in the HS sense, and conversely. In particular, if $X = \mathbf{R}$, the set of real numbers, then an HS integral is an HSL integral.

The following properties of the HSL integral can be easily verified.

Theorem 2.2. *Given any functions* $f, f_1\colon [a,b] \to \mathbf{R}$ *(resp.* X*),* $g, g_1\colon [a,b] \to X$ *(resp.* $\mathbf{R}$*), and any scalar* $\lambda \in \mathbf{R}$,

(*i*) *if f is* HSL-*integrable with respect to g on* $[a,b]$, *then f is* HSL-*integrable with respect to g on each interval* $[c,d]$ *in* $[a,b]$;

(*ii*) if $c \in (a,b)$, *and the integrals* $(HSL)\int_a^c f\,\mathrm{d}g$ *and* $(HSL)\int_c^b f\,\mathrm{d}g$ *exist, then* $(HSL)\int_a^b f\,\mathrm{d}g$ *exists, and*

$$(HSL)\int_a^b f\,\mathrm{d}g = (HSL)\int_a^c f\,\mathrm{d}g + (HSL)\int_c^b f\,\mathrm{d}g;$$

(*iii*) *if f and* f_1 *are* HSL-*integrable with respect to g on* $[a,b]$, *then the sum* $f+f_1$ *is* HSL-*integrable with respect to g on* $[a,b]$, *and*

$$(HSL)\int_a^b (f+f_1)\,\mathrm{d}g = (HSL)\int_a^b f\,\mathrm{d}g + (HSL)\int_a^b f_1\,\mathrm{d}g;$$

(*iv*) *if f is* HSL-*integrable with respect to both g and* g_1 *on* $[a,b]$, *then f is* HSL-*integrable with respect to* $g + g_1$ *on* $[a,b]$, *and*

$$(HSL)\int_a^b f\,\mathrm{d}(g+g_1) = (HSL)\int_a^b f\,\mathrm{d}g + (HSL)\int_a^b f\,\mathrm{d}g_1;$$

(*v*) *if f is* HSL-*integrable with respect to g on* $[a,b]$, *then so is* $\lambda\cdot f$, *and*

$$(HSL)\int_a^b (\lambda\cdot f)\,\mathrm{d}g = \lambda\cdot(HS)\int_a^b f\,\mathrm{d}g;$$

(*vi*) *if f is* HSL-*integrable with respect to g on* $[a,b]$, *then f is* HSL-*integrable with respect to* $\lambda\cdot f$, *and*

$$(HSL)\int_a^b f\,\mathrm{d}(\lambda\cdot g) = \lambda\cdot(HSL)\int_a^b f\,\mathrm{d}g.$$

Let us now turn to absolute integrability. For simplicity, we introduce the function

$$\|f\|(\xi) = \|f(\xi)\|$$

for all $\xi \in [a,b]$. Let g be a function of bounded variation on $[a,b]$. Then the function Vg_a given by

$$Vg_a(\xi) = Var(g;[a,\xi]),$$

for all $\xi \in [a,b]$, is a real-valued monotone increasing function on the interval $[a,b]$. If $f : [a,b] \to \mathbf{R}$ is continuous, then so is $|f|$. Then both f and $|f|$ are HS-integrable with respect to Vg_a on $[a,b]$. The following theorem gives necessary and sufficient conditions for the HS-integrability of $|f|$ with respect to Vg_a on $[a,b]$ in a general form.

Theorem 2.3 (Absolute Integrability Theorem). *Let $f : [a,b] \to X$ (resp. $\mathbf{R}$) be* HSL-*integrable with respect to $g : [a,b] \to \mathbf{R}$ (resp. X), where g is of bounded variation, with primitive F. Then $(HS)\int_a^b \|f\|\,\mathrm{d}Vg_a$ exists if and only if F is a function of bounded variation. Moreover,*

$$(HS)\int_a^b \|f\|\,\mathrm{d}Vg_a = Var(F;[a,b]).$$

Proof. We proceed with the case where we have $f : [a,b] \to X$ and $g : [a,b] \to \mathbf{R}$. We do the other case by interchanging norms.

($\Rightarrow$): For simplicity, put $\beta = Vg_a$. Let $x < y$ points in $[a,b]$, let $\varepsilon > 0$, and let $\delta(\xi) > 0$ be a function on $[a,b]$ for which

$$\left|(D)\sum \|f(\xi)\|\cdot\{\beta(v)-\beta(u)\} - (HS)\int_x^y |f|\,\mathrm{d}\beta\right| < \varepsilon$$

and it follows that

$$(D)\sum \left\|f(\xi)\{g(v)-g(u)\} - F(u,v)\right\| < \varepsilon$$

for all δ - fine divisions $D = \{([u,v]; \xi)\}$ of $[x,y]$. These inequalities imply that

$$\begin{aligned}\|F(x,y)\| - \varepsilon &\le (D)\sum\|F(u,v)\| - \varepsilon \\ &< (D)\sum\|f(\xi)\|\cdot|g(v)-g(u)| \\ &\le (D)\sum\|f(\xi)\|\cdot\{\beta(v)-\beta(u)\} \\ &< (HS)\int_x^y\|f\|\,d\beta + \varepsilon.\end{aligned}$$

Since ε is arbitrary, $\|F(x,y)\| \le (HS)\int_x^y\|f\|\,d\beta$. If $D = \{[x,y]\}$ is any partition of $[a,b]$, then

$$(D)\sum\|F(x,y)\| \le (D)\sum(HS)\int_x^y\|f\|\,d\beta = (HS)\int_a^b\|f\|\,d\beta.$$

This means that $Var(F;[a,b]) \le (HS)\int_a^b\|f\|\,d\beta$ and follows the desired result.

(⇐): Let $\varepsilon > 0$. There exists a partition $P = \{\, a = z_0, z_1, \ldots, z_n = b \,\}$ of $[a,b]$ such that if $[u,v]$ is a typical subinterval of the partition P, then

$$Var(F;[a,b]) - \varepsilon < (P)\sum\|F(v)-F(u)\| \le Var(F;[a,b]).$$

We note that the same inequalities hold if P is replaced by any partition $D \supset P$ of the interval $[a,b]$.

Similarly, for each integer $n > 0$, there exists a partition P_n of $[a,b]$ such that

$$Var(g;[a,b]) - \frac{\varepsilon}{n2^n} < (P_n)\sum\|g(v)-g(u)\| \le Var(g;[a,b]).$$

We shall assume $P \subset P_n$. Define $\delta_n = \min\{|v-u| : [u,v] \text{ is a subinterval of } P_n\}$. Also let

$$E_n = \{\, \xi \in [a,b] : n-1 \le \|f(\xi)\| < n \,\}$$

Notice that every $\xi \in [a,b]$ belongs to exactly one E_n and $[a,b] = \bigcup_n E_n$.

By definition of HSL-integrability, there exists a function $\delta^*(\xi) > 0$ on $[a,b]$ such that for all δ^*- fine divisions $D = \{([u,v];\xi)\}$,

$$(D)\sum\|F(v) - F(u) - f(\xi)\{g(v) - g(u)\}\| < \varepsilon.$$

Define for each $\xi \in [a,b]$, $\delta(\xi) = \min\left\{\delta^*(\xi), \frac{\delta_n}{2}\right\}$, where $\xi \in E_n$. We modify $\delta(\xi)$ so that $z_0, z_1, \ldots, z_n$ are associated points of every δ - fine division of the interval $[a,b]$.

Let $D = \{([u,v];\xi)\}$ be a δ - fine division of $[a,b]$. Construct a refinement Ξ of D as follows: If $u < z_j = \xi < v$, then $([u,\xi];\xi), ([\xi,v];\xi) \in \Xi$, and if $z_j = \xi$ is already an endpoint of $[u,v]$, where $([u,v];\xi) \in D$, then $([u,v];\xi) \in \Xi$. Hence,

$$Var(F;[a,b]) - \varepsilon < (\Xi)\sum\|F(v) - F(u)\| \le Var(F;[a,b]),$$

which implies that

$$\left|Var(F;[a,b]) - (\Xi)\sum\|F(v) - F(u)\|\right| < \varepsilon,$$

and

$$\left|(\Xi)\sum\|F(v) - F(u)\| - (\Xi)\sum\|f(\xi)\|\cdot|g(v) - g(u)|\right|$$

$$\le (\Xi)\sum\|F(v) - F(u) - f(\xi)\{g(v) - g(u)\}\| < \varepsilon.$$

For each n, let

$$\Xi_n = \{([u,v];\xi) \in \Xi : \xi \in E_n\}.$$

Then $\{\Xi_n\}$ partitions Ξ. For if $([u,v];\xi) \in \Xi_n \cap \Xi_m$, then we have $\xi \in E_n \cap E_m$ and $n = m$.

Now let $([u,v];\xi) \in \Xi_n$, and let $x^n_{j-1} \le \xi \le x^n_j$, for some adjacent points x^n_{j-1}, x^n_j in P_n. Since $x^n_j - x^n_{j-1} \ge \delta_n$, we have $[u,v] \subset [x^n_{j-1}, x^n_j]$ and thus

$$0 \le (\Xi_n)\sum\left\{(\beta(v) - \beta(u)) - |g(v) - g(u)|\right\}$$

$$\le (Q_n)\sum\left\{(\beta(v) - \beta(u)) - |g(v) - g(u)|\right\}$$

$$= Var(g;[a,b]) - (Q_n)\sum|g(v)-g(u)| < \frac{\varepsilon}{n2^n},$$

where Q_n is the partition of $[a,b]$ whose elements are members of the set $\{ u, v : ([u,v];\xi) \in \Xi_n \} \cup P_n$, thus a refinement of P_n. Therefore,

$$\left|(\Xi)\sum\|f(\xi)\|\{\beta(v)-\beta(u)\} - (\Xi)\sum\|f(\xi)\|\cdot|g(v)-g(u)|\right|$$

$$\leq (\Xi)\sum\|f(\xi)\|\cdot\{(\beta(v)-\beta(u)) - |g(v)-g(u)|\}$$

$$= \sum_n\left\{(\Xi_n)\sum\|f(\xi)\|\{(\beta(v)-\beta(u)) - |g(v)-g(u)|\}\right\}$$

$$< \sum_n\left\{(\Xi_n)\sum n\cdot\{(\beta(v)-\beta(u)) - |g(v)-g(u)|\}\right\}$$

$$< \sum_n n\cdot\frac{\varepsilon}{n2^n} = \varepsilon.$$

Applying the triangle inequality,

$$\left|(D)\sum\|f(\xi)\|\{\beta(v)-\beta(u)\} - Var(F;[a,b])\right| =$$

$$= \left|(\Xi)\sum\|f(\xi)\|\{\beta(v)-\beta(u)\} - Var(F;[a,b])\right|$$

$$\leq \left|(\Xi)\sum\|f(\xi)\|\{\beta(v)-\beta(u)\} - (\Xi)\sum\|f(\xi)\|\cdot|g(v)-g(u)|\right| +$$

$$+ \left|(\Xi)\sum\|f(\xi)\|\cdot|g(v)-g(u)| - (\Xi)\sum\|F(v)-F(u)\|\right| +$$

$$+ \left|(\Xi)\sum\|F(v)-F(u)\| - Var(F;[a,b])\right|$$

$$< 3\varepsilon. \quad \square$$

This next concept will help us find a necessary and sufficient condition for the existence of the HSL integral.

Definition 2.4. An interval function χ is said to be ***nonnegative*** if

$$\chi(x,y) \geq 0,$$

and ***superadditive*** if

$$\chi(x,y) + \chi(y,z) \leq \chi(x,z),$$

when real numbers x, y, and z satisfy the inequality $x < y < z$.

Theorem 2.5. *Let $f : [a,b] \to X$ (resp. $\mathbf{R}$) and $g : [a,b] \to \mathbf{R}$ (resp. X). Then f is HSL-integrable with respect to g on $[a,b]$ if and only if there exists a function $F : [a,b] \to X$ such that for every $\varepsilon > 0$, there is a positive function $\delta(\xi)$ on $[a,b]$ and there is a nonnegative superadditive interval function χ where*

$$\chi(a,b) < \varepsilon$$

and that whenever $\xi \in [u,v] \subset (\xi-\delta(\xi),\xi+\delta(\xi))$, we have

$$\|F(u,v) - f(\xi)\{g(v) - g(u)\}\| \leq \chi(u,v).$$

Proof. ($\Leftarrow$): Let $F : [a,b] \to X$ satisfy the given property. We show that F is the required function in Definition 2.1. Let $\varepsilon > 0$, and let $\delta(\xi)$ and χ be as in the hypothesis. If $D = \{([u,v];\ \xi)\}$ is a δ - fine division of $[a,b]$, then

$$(D)\sum\|F(u,v) - f(\xi)\{g(v) - g(u)\}\| \leq (D)\Sigma\chi(u,v) \leq \chi(a,b) < \varepsilon.$$

($\Rightarrow$): Let F denote the primitive of f with respect to g on $[a,b]$. Then, given $\varepsilon > 0$, there exists $\delta(\xi) > 0$ on $[a,b]$ such that for all divisions $D = \{([u,v];\xi)\}$ of the interval $[a,b]$,

$$(D)\sum\|F(u,v) - f(\xi)\{g(v) - g(u)\}\| < \frac{\varepsilon}{2}.$$

For each interval $[x,y] \subset [a,b]$, define χ by

$$\chi(x,y) = \sup (D)\sum\|F(u,v) - f(\xi)\{g(v) - g(u)\}\|,$$

where supremum is taken over all δ - fine divisions $D = \{([u,v];\xi)\}$ of $[x,y]$. Then χ is nonnegative superadditive satisfying the inequalities

$$\chi(a,b) \leq \frac{\varepsilon}{2} < \varepsilon. \quad \square$$

Next, we define absolutely summing operators.

Definition 2.6. Let $T : X \to Y$ be a bounded linear operator. T is said to be ***absolutely summing*** if there exists $\rho > 0$ such that for every finite set $\{ x_1, x_2, \ldots, x_n \} \subset X$, we have

$$\sum_{k=1}^{n} \|T(x_k)\| \le \rho \cdot \sup_{\substack{T^* \in X' \\ \|T^*\| \le 1}} \sum_{k=1}^{n} |T^*(x_k)|.$$

Theorem 2.7. *Let* $T : X \to Y$ *be absolutely summing and* $f : [a,b] \to X$ *(resp.* **R***) be* HS-*integrable with respect to* $g : [a,b] \to \mathbf{R}$ *(resp.* X*) on* $[a,b]$*. Then* Tf *(resp.* f*) is* HSL-*integrable with respect to* g *(resp.* Tg*) on* $[a,b]$.

Proof: Since T is absolutely summing, let the number $\rho > 0$ be as in Definition 2.6. Let F be a primitive of f with respect to g on $[a,b]$ (in the HS sense). By Theorem 1.3, corresponding to $\varepsilon > 0$ is a $\delta(\xi) > 0$ on $[a,b]$ such that given a δ - fine division $D = \{([u,v];\xi)\}$ of $[a,b]$,

$$\sup_{\substack{T \in X' \\ \|T\| \le 1}} (D)\sum \left\|T\left(F(v) - F(u) - f(\xi)\{g(v) - g(u)\}\right)\right\| < \frac{\varepsilon}{\rho}.$$

Consider the sequence $\{F(v) - F(u) - f(\xi)\{g(v) - g(u)\}\}_D$. We have

$$(D)\sum \|TF(v) - TF(u) - Tf(\xi)\{g(v) - g(u)\}\| =$$

$$= (D)\sum \left\|T\left(F(v) - F(u) - f(\xi)\{g(v) - g(u)\}\right)\right\|$$

$$\le \rho \cdot \sup_{\substack{T^* \in X' \\ \|T^*\| \le 1}} (D)\sum \left|T^*\left(F(v) - F(u) - f(\xi)\{g(v) - g(u)\}\right)\right|$$

$$< \varepsilon.$$

Therefore, Tf is HSL-integrable with respect to g on $[a,b]$ with primitive TF.

The second result follows from the fact that

$$(D)\sum \|TF(v) - TF(u) - f(\xi)\{Tg(v) - Tg(u)\}\| =$$

$$= (D)\sum \left\|T\left(F(v) - F(u) - f(\xi)\{g(v) - g(u)\}\right)\right\|. \quad \square$$

In Theorem 2.7, if T is the identity operator $1 : X \to X$, then an HS integral is an HSL integral. Accordingly, if $1: X \to X$ is absolutely summing, then X has a finite dimension. This last theorem will show that indeed in a finite dimensional space X, the integrals HS and HSL coincide.

Theorem 2.8. *Suppose X has a finite dimension, say n. Then $f : [a,b] \to X$ (resp. $\mathbf{R}$) is* HS-*integrable with respect to $g : [a,b] \to \mathbf{R}$ (resp. X) on $[a,b]$ if and only if f is* HSL-*integrable with respect to g on $[a,b]$.*

Proof. We complete the proof by showing that if $f: [a,b] \to X$ (resp. $\mathbf{R}$) is HS-integrable with respect to $g : [a,b] \to \mathbf{R}$ (resp. X) on $[a,b]$, then f is HSL-integrable with respect to g on $[a,b]$. Now we take up the case where we have the functions $f: [a,b] \to X$ and $g : [a,b] \to \mathbf{R}$. By the isomorphism of the Banach spaces X and $\mathbf{R}^n$, it is enough to take a HS-integrable $f: [a,b] \to \mathbf{R}^n$ with respect to $g : [a,b] \to \mathbf{R}$ on $[a,b]$. Let f be given by

$$f(t) = (f_1(t), f_2(t), \ldots, f_n(t)),$$

for some real-valued functions f_i, $i = 1, 2, \ldots, n$, on $[a,b]$.

In $\mathbf{R}^n$, all norms are equivalent, and we may use the norm

$$\|w\| = \sum_{i=1}^{n} |x_i|, \text{ for all } w = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n.$$

Since the absolute value of a component of a vector does not exceed the norm of the vector, Cauchy criterion implies that each f_i must be HS-integrable with respect to g on $[a,b]$. By Henstock's lemma, the integral $(HSL)\int_a^b f_i dg$ exists, for all $i = 1, 2, \ldots, n$. Let F_i denote the primitive of f_i with respect to g on $[a,b]$, and set

$$F(t) = (F_1(t), F_2(t), \ldots, F_n(t))$$

for all $t \in [a,b]$. Given $\varepsilon > 0$, let $\delta(\xi) > 0$ on $[a,b]$ be the common required function in Definition 2.1 for the HSL-integrability of the functions f_i with respect to g on $[a,b]$. If $D = \{([u,v];\xi)\}$ is a δ - fine division of $[a,b]$, then

$$(D)\sum \|F(u,v) - f(\xi)\{g(v) - g(u)\}\| =$$

$$= (D)\sum\left\{\sum_{i=1}^{n}\left|F_i(u,v) - f(\xi)\{g(v) - g(u)\}\right|\right\}$$

$$= \sum_{i=1}^{n}\left\{(D)\sum\left|F_i(u,v) - f(\xi)\{g(v) - g(u)\}\right|\right\}$$

$$< n\cdot\varepsilon.$$

Since n is fixed and ε is arbitrary, the proof is complete. □

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