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An Integration by Parts Formula For an Integral in Local System

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Abstract

In this paper, we give an integration by parts formula for the S*-integral introduced by Wang and Ding* [1]. *Also, we will show that a real-valued function of bounded variation can be a multiplier for the* S*-integrable functions.*

Some integration by parts formulas for the Henstock integral as well as for the ASP-integral had been proved (See [1] and [3]). Such formulas had been successfully obtained by first introducing Stieltjes-type integrals. Lee in [1] defined the Henstock-Stieltjes integral (which turned out to be equivalent to the Perron-Stieltjes integral) and Xu Dongfu, et. al. [3] introduced the ASP-Stieltjes integral.

In this paper, we define a Stieltjes-type integral using Thomson's local system. We shall then use this new concept to obtain an integration by parts formula for the S-integral introduced by Wang and Ding [2]. Moreover, we will show that a function of bounded variation can be a multiplier for the S-integrable functions.

1. Preliminaries

Definition 1.1. Let **R** be the set of real numbers. Suppose for every $x \in \mathbf{R}$ there corresponds a non-empty family $S(x)$ of subsets of **R** satisfying the following conditions:

(*i*) $\{x\} \notin S(x)$;

(*ii*) if $\sigma \in S(x)$, then $x \in \sigma$;

(*iii*) if $\sigma_1 \in S(x)$ and $\sigma_1 \subset \sigma_2$, then $\sigma_2 \in S(x)$;

(*iv*) if $\sigma \in S(x)$ and $\delta > 0$, then $\sigma \cap (x - \delta, x + \delta) \in S(x)$.

Then $S = \{S(x) : x \in \mathbf{R}\}$ *is called* a ***local system***.

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Definition 1.2. A local system S is said to be ***bilateral*** if every set $\sigma \in S(x)$ contains points on either side of x. It is said to be ***filtering*** if for every x, we have $\sigma_1 \cap \sigma_2 \in S(x)$ whenever σ_1 and σ_2 belong to $S(x)$. It satisfies the ***intersection condition*** if for every collection of sets $\{\sigma_x : x \in \mathbf{R}\}$ with $\sigma_x \in S(x)$ there exists a positive function δ such that if $0 < y - x < \min\{\delta(y), \delta(x)\}$, then $\sigma_x \cap \sigma_y \cap [x,y] \neq \emptyset$. The set $\{\sigma_x : x \in \mathbf{R}\}$ is called a ***choice*** from S.

Definition 1.3. A family C of intervals is called an S-***cover*** of $\mathbf{R}$ if for each point $x \in \mathbf{R}$ the set

$$\sigma(x) = \{y : y = x, \text{ or } y > x \text{ and } [x,y] \in C, \text{ or } y < x \text{ and } [y,x] \in C\}$$

belongs to the family $S(x)$. The family $\{\sigma(x) : x \in \mathbf{R}\}$ is a ***choice*** from S.

Lemma 1.4 (Intersection Lemma). *Suppose that the local system is filtering. If C_1 and C_2 are* S*-covers of* $\mathbf{R}$, *then so is* $C_1 \cap C_2$.

Lemma 1.5 (Thomson's Lemma). *Let* S *be a local system, which is bilateral and has the intersection condition. If C is an* S*-cover of* $\mathbf{R}$, *then there exists a C-partition* $D = \{[u,v]\}$ *of any interval* $[a,b]$.

Throughout, we have the following definition for an associated point (tag) ξ of $[u,v]$ in a division $D = \{[u,v]\}$ of $[a,b]$.

Definition 1.6. Let $D = \{[u,v]\}$ be a division of an S-cover C of $\mathbf{R}$ corresponding to the choice $\{\sigma(x) : x \in \mathbf{R}\}$ from S. Then the ***associated point*** ξ of $[u,v]$ is defined to be u, if $v \in \sigma(u)$, and v, if $u \in \sigma(v)$.

Definition 1.7. A function $f : [a,b] \to \mathbf{R}$ is said to be S-integrable to the number A if for every $\varepsilon > 0$ there exists an S-cover C of $\mathbf{R}$ such that for any C-partition $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$\left|(D)\sum f(\xi)(v-u) - A\right| < \varepsilon.$$

In this case we write $(S)\int_a^b f(x)dx = A$.

Theorem 1.8 (Henstock's Lemma). *Let* $f : [a,b] \to \mathbf{R}$ *be* S*-integrable and let* $F(x) = (S)\int_a^x f(t)dt$ be its primitive. *Then for every* $\varepsilon > 0$, *there exists an* S*-cover C of* $\mathbf{R}$ *such that for any* C*-partition* $D = \{([u,v];\xi)\}$ *of* $[a,b]$, *we have*

$$(D)\sum \left|f(\xi)(v-u) - F(v) + F(u)\right| < \varepsilon.$$

Definition 1.9. A function $F : [a,b] \to \mathbf{R}$ is said to be S-***continuous*** at the point x if for every $\varepsilon > 0$ there exists a $\sigma(x) \in S(x)$ such that for every $y \in \sigma(x) \cap [a,b]$, we have $|F(x) - F(y)| < \varepsilon$.

Definition 1.10. A function $f : [a,b] \to \mathbf{R}$ is said to be S-***Stieltjes integrable*** with respect to a real-valued function g on $[a,b]$ to a number J if for every $\varepsilon > 0$, there exists an S-cover C of $\mathbf{R}$ such that for every C-partition $D = \{([u,v];\xi)\}$ of $[a,b]$, we have $|(D)\ \Sigma f(\xi)\{g(v) - g(u)\} - J| < \varepsilon$. In this case we write,

$$J = (SS)\int_a^b f\,dg \quad \text{or} \quad J = \int_a^b f\,dg, \text{ if there is no ambiguity.}$$

2. Results

Theorem 2.1. *If* $f : [a,b] \to \mathbf{R}$ *is* S-*integrable on* $[a,b]$ *with* S-*primitive* F *given by*

$$F(x) = (S)\int_a^x f(t)dt,$$

then F *is* S-*continuous on* $[a,b]$.

Proof. Let $z \in [a,b]$ and let $\varepsilon > 0$. Then, by Henstock's Lemma for the S-integral, there exists an S-cover C of $\mathbf{R}$ such that for any C-partition $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$(D)\Sigma|f(\xi)(v - u) - F(v) + F(u)| < \frac{\varepsilon}{2}.$$

Let $\{\sigma(x)\}$ be the choice in S belonging to the S-cover C. Let

$$\tau(z) = \sigma(z) \cap \left(z - \frac{\varepsilon}{4(1+|f(z)|)}, z + \frac{\varepsilon}{4(1+|f(z)|)}\right).$$

Then $\tau(z) \in S(z)$. Now, let $y \in \tau(z)$. Then either $[y,z]$ or $[z,y] \in C$. Thus

$$|F(z) - F(y)| \le |F(z) + F(y) - f(z)(z - y)| + |f(z)(z - y)|$$

$$< \frac{\varepsilon}{2} + (1 + |f(z)|).\frac{\varepsilon}{2(1+|f(z)|)} = \varepsilon.$$

Therefore, F is S-continuous on $[a,b]$. □

The proof of the next theorem follows from a standard argument.

Theorem 2.2. *If the S-Stieltjes integral of f with respect to g on [a,b] exists, then it is unique.*

Theorem 2.3 (Integration by Parts). *Let $F : [a,b] \to \mathbf{R}$ be S-continuous on [a,b] and $g : [a,b] \to \mathbf{R}$ be of bounded variation on [a,b]. Then $\int_a^b F\,dg$ exists if and only if $\int_a^b g\,dF$ exists. Moreover, if $\int_a^b F\,dg$ or $\int_a^b g\,dF$ exists, then*

$$\int_a^b F\,dg + \int_a^b g\,dF = F(b)g(b) - F(a)g(a).$$

Proof. We prove that the condition is necessary; sufficiency follows by using a similar argument. Suppose that $\int_a^b F\,dg$ exists. Then, given $\varepsilon > 0$, there exists an S-cover C_1 of $\mathbf{R}$ such that for any C_1-partition $D_1 = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$\left|(D)\sum F(\xi)\{g(v) - g(u)\} - \int_a^b F\,dg\right| < \frac{\varepsilon}{3}.$$

Since F is S-continuous on $[a,b]$, for each $x \in [a,b]$, there exists a $\sigma(x) \in S(x)$ such that whenever $y \in \sigma(x)\cap[a,b]$, then

$$|F(x) - F(y)| < \frac{\varepsilon}{3(1+V(g;[a,b])}.$$

Let $C_2 = \{[u,v], [v,u] : u \in \sigma(v) \text{ or } v \in \sigma(u)\}$. Then C_2 is an S-cover of $\mathbf{R}$. Let $C = C_1\cap C_2$. Then, by the Intersection Lemma, C is an S-cover of $\mathbf{R}$. Let D $= \{([u,v];\xi)\}$ be a C-partition of $[a,b]$. Then D is both a C_1 and a C_2-partition of $[a,b]$. Thus

$$\left|(D)\sum g(\xi)\{F(v) - F(u)\} - F(b)g(b) + F(a)g(a) + \int_a^b F\,dg\right| =$$

$$= \left|(D)\sum \{g(\xi)[F(v) - F(u)] - F(v)g(v) + F(u)g(u)\} + \int_a^b F\,dg\right|$$

$$\leq \left|(D)\sum \{(g(\xi) - g(v))(F(v) - F(\xi)) - (g(\xi) - g(u))(F(u) - F(\xi))\}\right| +$$

$$+ \left|(D)\sum \{F(\xi)(g(\xi) - g(v) - g(\xi) + g(u))\} - \int_a^b F\,dg\right|$$

$$\leq (D)\sum |g(\xi) - g(v)||F(v) - F(\xi)| + (D)\sum |g(\xi) - g(u)||F(u) - F(\xi)|$$

$$+ \left|(D)\sum F(\xi)\{g(v) - g(u)\} - \int_a^b F\,dg\right|$$

$$< \frac{\varepsilon}{3(1+V(g))}(1+V(g)) + \frac{\varepsilon}{3(1+V(g))}(1+V(g)) + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore, g is S-Stieltjes integrable with respect to F on $[a,b]$ and

$$\int_a^b F\,dg + \int_a^b g\,dF = F(b)g(b) - F(a)g(a). \quad \square$$

We will use the preceding theorem to prove the following theorem.

Theorem 2.4. *If* $f : [a,b] \to \mathbf{R}$ *is* S-*integrable with* S-*primitive* F, *and the function* $g : [a,b] \to \mathbf{R}$ *is of bounded variation, and if* $\int_a^b F dg$ *exists, then* fg *is* S-*integrable and*

$$(S)\int_a^b f(x)g(x)\,dx = F(b)g(b) - F(a)g(a) - \int_a^b F\,dg.$$

Proof. Without loss of generality, we may assume that $F(a) = 0$. Let $\varepsilon > 0$. Then, by Henstock's Lemma for S-integrals, there exists an S-cover C_1 of **R** such that if $D_1 = \{([u,v];\xi)\}$ is a C_1-partition of $[a,b]$, then

$$(D_1)\sum |f(\xi)(v-u) - F(v) + F(u)| < \varepsilon.$$

By Theorem 2.3, $\int_a^b g\,dF$ exists, and hence there exists an S-cover C_2 of **R** such that if $D_2 = \{([u,v];\xi)\}$ is a C_2-partition of $[a,b]$, then

$$\left|(D)\sum g(\xi)\{F(v) - F(u)\} - \int_a^b g\,dF\right| < \varepsilon.$$

Let $C = C_1 \cap C_2$. Then, by the Intersection Lemma, C is an S-cover of **R**. Let $\{\sigma(x) : x \in \mathbf{R}\}$ be the choice from S determined by C. For each $x \in \mathbf{R}$, define $\tau(x)$ to be $\sigma(x) \cap (b,\infty)$ if $x > b$, $\sigma(x)$ if $x = b$, and $\sigma(x) \cap (-\infty,b)$ if $x < b$. Then the set $\{\tau(x) : x \in \mathbf{R}\}$ is a choice from S. Let C^* be an S-cover of **R** corresponding to the choice $\{\tau(x) : x \in \mathbf{R}\}$ and let D be a C^*-partition of $[a,b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b \quad \text{and} \quad \{\xi_1, \xi_2, \xi_3, \ldots, \xi_n\}.$$

Then $\xi_n = b$. Thus,

$$\left| \sum_{k=1}^{n} g(\xi_k) f(\xi_k)(x_k - x_{k-1}) - \int_a^b g\, dF \right|$$

$$= \Big| g(\xi_1) f(\xi_1)(x_1 - x_0) + \Big\{ \sum_{k=2}^{n} g(\xi_k) \Big[\sum_{i=1}^{k} f(\xi_i)(x_i - x_{i-1})$$

$$- \sum_{i=1}^{k-1} f(\xi)(x_i - x_{i-1}) \Big] \Big\} - \int_a^b g\, dF \Big|$$

$$= \Big| \sum_{k=1}^{n-1} [g(\xi_k) - g(\xi_{k+1})] \Big[\sum_{i=1}^{k} f(\xi_i)(x_i - x_{i-1}) - F(x_k) \Big] \Big|$$

$$\Big| g(b) \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - g(b)F(b) \Big| +$$

$$+ \Big| \sum_{k=1}^{n} g(\xi_k)[F(x_k - F(x_{k-1})] - \int_a^b g\, dF \Big|$$

$$< 2\varepsilon\, V(g; [a,b]) + |g(b)|\, \varepsilon + \varepsilon .$$

Thus, fg is S-integrable on $[a,b]$ and

$$(\mathrm{S}) \int_a^b f(t) g(t)\, dt = \int_a^b g\, dF.$$

The result now follows from Theorem 2.3. □

References

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