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The Cauchy Extension Theorem for Bilinear Henstock-Stieltjes Integrals

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Abstract

In this paper, we state and prove the Cauchy extension theorem for the bilinear Henstock-Stieltjes integral and cite particular applications.

1. Introduction

Throughout this paper, we shall always use X, Y, and Z to denote real Banach spaces. The symbol $L(X, Y; Z)$ is used to denote the space of all bounded bilinear transformations $A : X \times Y \to Z$.

Let $A \in L(X, Y; Z)$, $f : [a,b] \to X$ and $g : [a,b] \to Y$ be functions. We say that f is ***Henstock-Stieltjes*** integrable (HS-***integrable***) with respect to A and g on $[a,b]$ if there is a vector J in Z satisfying the following condition: For every $\varepsilon > 0$, there exists a positive function $\delta(\xi)$ defined on $[a,b]$ such that for any δ-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$,

$$\left\|(D)\sum A(f(\xi),\ g(v) - g(u)) - J\right\| < \varepsilon.$$

In this case we write

$$J = (HS)\int_a^b A(f,\ dg).$$

Recall that a division $D = \{([u,v];\xi)\}$ is δ-***fine*** if $\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ for every interval-point pair $([u,v];\xi)$ in D. Any subset of D is a δ-fine partial division of $[a,b]$.

The above integral is called ***bilinear Henstock-Stieltjes integral***. As always, we put $(HS)\int_a^a A(f,\ dg) = 0$.

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Canoy [1] initiated the work on bilinear Henstock-Stieltjes integral. More results have been obtained in [2], [3], [4], and [7]. The existence of the integral $(HS)\int_a^b A(f,dg)$ has been proved in the case where f and g are regulated and VB^* (see [3], and [4]). It is the intention of the present paper to prove the Cauchy extension theorem for the above integral. Examples will be given to illustrate the usefulness of the theorem.

Results

Throughout this study all integrals are, unless otherwise specified, Henstock-Stieltjes integrals. The following two lemmas are needed to prove the Cauchy extension theorem. The first is a version of Henstock's lemma.

Lemma 1. *Let $A \in L(X, Y; Z)$, and suppose that $f : [a,b] \to X$ is HS-integrable with respect to $g : [a,b] \to Y$. Define $F(x) = \int_a^x A(f,\ dg)$ for all x in $[a,b]$. Given $\varepsilon > 0$, there exists a positive function $\delta(\xi) > 0$ on $[a,b]$ such that for all δ-fine partial divisions $D = \{([u,v];\xi)\}$ of $[a,b]$, we have*

$$\left\|(D)\sum\{F(u,v) - A(f(\xi),\ g(v)\text{-}g(u))\}\right\| < \varepsilon.$$

Proof. Given $\varepsilon > 0$, there exists a positive function $\delta(\xi) > 0$ on $[a,b]$ such that for all δ-fine divisions $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$\left\|(D)\sum\{F(u,v) - A(f(\xi),\ g(v)\text{-}g(u))\}\right\| < \frac{\varepsilon}{2}.$$

Let D be a δ-fine division of $[a,b]$ and Σ_1 a partial sum of Σ when Σ takes over D. Set E_1 to be the union of the intervals $[u,v]$ from Σ_1, and let E_2 define the closure of $[a,b] \setminus E_1$. Then E_2 is the union of m disjoint closed intervals $[a_i,b_i]$. Since each $\int_{a_i}^{b_i} A(f,\ dg)$ exists, there is $0 < \delta^*(\xi) \le \delta(\xi)$ on E_2 such that if D_i is a δ^*-fine division of $[a_i,b_i]$, then

$$\left\|(D_i)\sum\{F(u,v) - A(f(\xi),\ g(v) - g(u))\}\right\| < \frac{\varepsilon}{2m},$$

$i = 1, 2, \ldots, m$. Now $D^* = E_1 \cup (\bigcup_{i=1}^{m} D_i)$ is a δ-fine division of $[a,b]$ and

$$(D^*)\sum A(f(\xi),\, g(v)-g(u)) = \sum_1 A(f(\xi),\, g(v)-g(u)) +$$

$$+\sum_{i=1}^{m}\Big((D_i)\sum A(f(\xi),\, g(v)-g(u))\Big).$$

Therefore,

$$\Big\|\sum_1 \{F(u,v)-A(f(\xi),\, g(v)-g(u))\}\Big\| \leq$$

$$\leq \Big\|(D^*)\sum \{F(u,v)-A(f(\xi),\, g(v)-g(u))\}\Big\| +$$

$$+\sum_{i=1}^{m}\Big\{\Big\|(D_i)\sum \{F(u,v)-A(f(\xi),\, g(v)-g(u))\}\Big\|\Big\}$$

$$< \frac{\varepsilon}{2} + \sum_{i=1}^{m}\frac{\varepsilon}{2m} = \varepsilon,$$

and this proves the theorem. □

Lemma 2. *Let $A \in L(X, Y; Z)$, $f : [a,b] \to X$, and $g : [a,b] \to Y$.*

(*i*) *Suppose that the integral $\int_x^b A(f,\, dg)$ exists for every $x \in (a,b]$. For every $\varepsilon > 0$, there exists a positive function $\delta(\xi)$ on $(a,b]$ such that if $c \in (a,b)$ and $D = \{([u,v];\xi)\}$ is any δ - fine division of $[c,b]$, then*

$$\Big\|(D)\sum A(f(\xi),\, g(v)-g(u)) - \int_c^b A(f,\, dg)\Big\| < \varepsilon.$$

(*ii*) *Suppose that the integral $\int_a^x A(f,\, dg)$ exists for every $x \in [a,b)$. For every $\varepsilon > 0$, there exists a positive function $\delta(\xi)$ on $[a,b)$ such that if $c \in (a,b)$ and $D = \{([u,v];\xi)\}$ is any δ - fine division of $[a,c]$, then*

$$\Big\|(D)\sum A(f(\xi),\, g(v)-g(u)) - \int_a^c A(f,\, dg)\Big\| < \varepsilon.$$

Proof. (*i*) Let $\{a_n\}$ be a decreasing sequence of points in (a,b) that converges to a. Set $b = a_0$ and let $\varepsilon > 0$. By Lemma 1, for each $n \geq 1$, there exists a positive function $\delta_n(\xi)$ on $[a_n, a_{n-1}]$ such that for all δ_n - fine partial divisions $D = \{([u,v];\, \xi)\}$ of $[a_n, a_{n-1}]$,

$$\left\|(D)\sum\left\{A(f(\xi),\, g(v)-g(u)) \;-\; \int_u^v A(f,\, dg)\right\}\right\| \;<\; \frac{\varepsilon}{2^n}.$$

Define on $(a,b]$ the function

$$\delta(\xi)=\begin{cases}\min\{\delta_1(\xi), a_0-a_1\}, \text{ if } \ \xi=a_0,\\ \min\{\delta_n(\xi), a_{n-1}-\xi, \xi-a_n\}, \text{if } \ a_n<\xi<a_{n-1} \text{ for some } \ n\geq 1,\\ \min\{\delta_n(\xi), \delta_{n+1}(\xi), a_n-a_{n+1}, a_{n-1}-a_n\}, \text{if } \ \xi=a_n \ \text{ for some } \ n\geq 1.\end{cases}$$

Let $c\in(a,b)$ and $D=\{([u,v];\xi)\}$ be a δ_n - fine division of $[c,b]$. Since

$$A(f(\xi),\, g(v)-g(u)) = A(f(\xi),\, g(v)-g(\xi)) + A(f(\xi),\, g(\xi)-g(u)),$$

we can assume that every associated point is an endpoint. There exists a positive integer m where $a_{m+1} < c \leq a_m$. By definition of $\delta(\xi)$, a_n is an associated point in D for every $n = 0, 1, 2, \ldots, m$. Further, for every interval-point pair $([u,v];\, \xi) \in D$, there is some $n = 1, 2, \ldots, m+1$ where $[u,v]\subset[a_n, a_{n-1}]$. For any such n, let D_n denote the set of pairs $([u,v];\, \xi)$ in D such that $[u,v]\subset[a_n,a_{n-1}]$. Then D_n is a δ_n - fine partial division of $[a_n, a_{n-1}]$. Note that $D_{m+1} = D\setminus\bigcup_1^m D_n$ is a δ_{m+1} - fine partial division of $[a_{m+1}, a_m]$. Therefore,

$$\left\|(D)\sum A(f(\xi),\, g(v)-g(u)) \;-\; \int_c^b A(f,\, dg)\right\| \;=$$

$$=\left\|(D)\sum\left\{A(f(\xi),\, g(v)-g(u)) \;-\; \int_u^v A(f,\, dg)\right\}\right\|$$

$$\leq\sum_{n=1}^{m+1}\left\|(D_n)\sum\left\{A(f(\xi),\, g(v)-g(u)) \;-\; \int_u^v A(f,\, dg)\right\}\right\|$$

$$<\sum_{n=1}^{m+1}\frac{\varepsilon}{2^n} \;\leq\; \varepsilon.$$

Part (ii) is done similarly. □

Theorem 3 (Cauchy Extension). *Let $A \in L(X, Y; Z)$ and let $g : [a,b] \to Y$ and $f : [a,b] \to X$.*

(i) If $g(a+)$ and the integral $\int_x^b A(f, \mathrm{d}g)$ exist for every $x \in (a,b]$, then the integral $\int_a^b A(f, \mathrm{d}g)$ exists if and only if $\lim_{x \to a^+} \int_x^b A(f, \mathrm{d}g)$ exists. Moreover,

$$\int_a^b A(f, \mathrm{d}g) = A(f(a), g(a+) - g(a)) + \lim_{x \to a^+} \int_x^b A(f, \mathrm{d}g).$$

(ii) If $g(b-)$ and the integral $\int_a^x A(f, \mathrm{d}g)$ exists for every $x \in [a,b)$, then the integral $\int_a^b A(f, \mathrm{d}g)$ exists if and only if $\lim_{x \to b^-} \int_a^x A(f, \mathrm{d}g)$ exists. Moreover,

$$\int_a^b A(f, \mathrm{d}g) = A(f(b), g(b) - g(b-)) + \lim_{x \to b^-} \int_a^x A(f, \mathrm{d}g).$$

Proof. (*i*) Suppose $\int_a^b A(f, \mathrm{d}g)$ exists. The case where $A = 0$ is trivial. Given $\varepsilon > 0$, let $\delta_1 > 0$ be such that if $0 < x - a < \delta_1$, then

$$\|g(x) - g(a+)\| < \frac{\varepsilon}{\|A\|}.$$

By Lemma 1, there exists $0 < \delta_2(\xi) \leq \delta_1$ on $[a,b]$ such that for all δ_2 - fine partial divisions $D = \{([u,v];\xi)\}$ of $[a,b]$,

$$\left\|(D)\sum\left\{A(f(\xi), g(v) - g(u)) - \int_u^v A(f, \mathrm{d}g)\right\}\right\| < \frac{\varepsilon}{2}.$$

Let $0 < x - a < \delta_2(a)$. Take a δ_2 - fine division $D = \{([u,v];\xi)\}$ of $[x,b]$. Then $D^* = D \cup \{([a,x];a)\}$ is a δ_2 - fine division of $[a,b]$, and

$$\left\|\int_x^b A(f, \mathrm{d}g) - \int_a^b A(f, \mathrm{d}g) + A(f(a), g(a+) - g(a))\right\| \leq$$

$$\leq \left\|(D)\sum\left\{A(f(\xi), g(v) - g(u)) - \int_u^v A(f, \mathrm{d}g)\right\}\right\| +$$

$$+ \|A(f(a), g(a+) - g(a)) - A(f(a), g(x) - g(a))\| +$$

$$+\left\|(D^*)\sum\left\{A(f(\xi),g(v)-g(u))-\int_u^v A(f,\mathrm{d}g)\right\}\right\|$$

$$<\ \varepsilon+\|A\|\cdot\|f(a)\|\cdot\|g(a+)-g(x)\|\ <\ \varepsilon\cdot\{1+\|f(a)\|\}.$$

This means that

$$\lim_{x\to a^+}\int_x^b A(f,\mathrm{d}g)=\int_a^b A(f,\mathrm{d}g)\ -A(f(a),g(a+)-g(a)).$$

Conversely, consider the function

$$G(x)\ =\ \begin{cases}0 & ,\ x=a,\\ \int_x^b A(f,\ \mathrm{d}g), & x\neq a\end{cases}$$

on the interval $[a,b]$, and let $G(a+)=\ y\in Z$. Given $\varepsilon>0$, there exists $\delta_1>0$ such that if $0<\ x-a\ <\delta_1$, then

$$\|G(x)-y\|<\varepsilon\ \text{and}\ \|g(x)-g(a+)\|<\varepsilon.$$

Let $\delta_2(\xi)$ be as the function $\delta(\xi)$ on $(a,b]$ in Lemma 2. Define

$$\delta(a)=\frac{\delta_1}{2}\ \text{and}\ \delta(\xi)=\min\{\delta_2(\xi),\ \xi-a\}\text{for}\ \xi\in(a,b].$$

Let

$$D:\ a=x_0<x_1<\cdots<x_n=b,\ \{\xi_1,\xi_2,\ldots,\xi_n\}$$

be a δ - fine division of $[a,b]$, and set

$$D_1:x_1<x_2<\cdots<x_n=b,\ \{\xi_2,\xi_3,\ldots,\xi_n\}$$

Then a is an associated point of D and

$$\left\|(D)\sum A(f(\xi),\ g(v)-g(u))\ -y-A(f(a),\ g(a+)-g(a))\right\|$$

$$\leq\ \|A(f(a),\ g(x_1)-g(a))-A(f(a),\ g(a+)-g(a))\|\ +$$

$$+ \left\|(D_1)\sum A(f(\xi), g(v) - g(u)) - G(x_1)\right\| + \left\|G(x_1) - y\right\|$$

$$< \left\|A(f(a), g(x_1) - g(a+))\right\| + 2\varepsilon$$

$$< \varepsilon \cdot \left\{2 + \|f(a)\| \cdot \|A\|\right\}.$$

Since ε is arbitrary, we have

$$\int_a^b A(f, \mathrm{d}g) = A(f(a), g(a+) - g(a)) + \lim_{x \to a^+} \int_x^b A(f, \mathrm{d}g).$$

A parallel proof proves (*ii*). □

In what follows, we shall see that in some instances, proofs are much easier to carry out when Cauchy extension theorem is used. The proofs of the first and the third of the next four results were already given in [3] but directly from the definition of the integral.

Corollary 4. *Let $A \in L(X, Y; Z)$, and let $c \in [a,b]$. Let $f: [a,b] \to X$ be a function with $f(\xi) = 0$ for all $\xi \neq c$. If $g : [a,b] \to Y$ has one-sided limits at c, then $\int_a^b A(f, \mathrm{d}g)$ exists, and*

$$\int_a^b A(f, \mathrm{d}g) = A(f(c), g(c+) - g(c-)).$$

Proof. We set $g(a-) = g(a)$ and $g(b+) = g(b)$. Suppose that $c = a$. Since $\lim_{x \to a^+} \int_x^b A(f, \mathrm{d}g) = 0$, the conclusion follows from Theorem 3(*i*). Similarly, Theorem 3(*ii*) proves the case where $c = b$. The case where $a < c < b$ can be done using these two results and the linearity of the integral over subintervals. □

Corollary 5. *Let $A \in L(X, Y; Z)$, and let $c \in [a,b]$. If $g : [a,b] \to Y$ is a function with $g(\xi) = 0$ for all $\xi \neq c$, then $\int_a^b A(f, \mathrm{d}g)$ exists for any function $f: [a,b] \to X$.*

Proof. If $c = a$, then Theorem 3(i) implies $\int_a^b A(f, \mathrm{d}g) = -A(f(a), g(a))$. If $c = b$, then Theorem 3(ii) implies $\int_a^b A(f, \mathrm{d}g) = A(f(b), g(b)$. By these two results, $\int_a^b A(f, \mathrm{d}g) = 0$ whenever c is different from a and b. □

Corollary 6. *Let $A \in L(X, Y; Z)$, let $z \in X$, let $I = (c,d) \subset [a,b]$, and let f be given by*

$$f(\xi) = \begin{cases} z, & \xi \in I, \\ 0, & \text{otherwise.} \end{cases}$$

For any function $g : [a,b] \to Y$, if $g(c+)$ and $g(d-)$ exist, then $\int_a^b A(f, \mathrm{d}g)$ exists, and

$$\int_a^b A(f, \mathrm{d}g) = A(z, g(d-) - g(c+)).$$

Proof. We have

$$\lim_{x \to c^+} \int_x^e A(f, \mathrm{d}g) = A(z, g(e) - g(c+)) \text{ and}$$

$$\lim_{x \to d^-} \int_e^x A(f, \mathrm{d}g) = A(z, g(d-) - g(e)),$$

where $c < e < d$. Using Theorem 3 and the linearity of the integral over subintervals, we have

$$\int_a^b A(f, \mathrm{d}g) = A(z, g(e) - g(c+)) + A(z, g(d-) - g(e))$$

$$= A(z, g(d+) - g(c+)). \quad \square$$

Corollary 7. *Let $A \in L(X, Y; Z)$, let $z \in Y$, let $I = (c,d) \subset [a,b]$, and let a function g on $[a,b]$ be given by*

$$g(\xi) = \begin{cases} z, & \xi \in I, \\ 0, & \text{otherwise.} \end{cases}$$

For any function $f : [a,b] \to X$, $\int_a^b A(f, dg)$ exists, and

$$\int_a^b A(f, dg) = A(f(c) - f(d), z).$$

Proof. Note that

$$\lim_{x \to c^+} \int_x^e A(f, dg) = \lim_{x \to d^-} \int_e^x A(f, dg) = \int_a^c A(f, dg) = \int_d^b A(f, dg) = 0,$$

where $c < e < d$. By Theorem 3 and the linearity of the integral over subintervals, we have

$$\int_a^b A(f, dg) = A(f(c), z) + A(f(d), -z) = A(f(c) - f(d), z). \quad \square$$

Theorem 8. *Let $A \in L(X, Y; Z)$ and $a = x_0 < x_1 < \ldots < x_n = b$. Let $g : [a,b] \to Y$ be a step function with $g(\xi) = a_j$ when $x_{j-1} < \xi < x_j$. For any function $f : [a,b] \to X$, the integral $\int_a^b A(f, dg)$ exists, and*

$$\int_a^b A(f, dg) = \sum_{i=1}^{n} \left\{ A(f(x_{i-1}), a_i - g(x_{i-1})) + A(f(x_i), g(x_i) - a_i) \right\}.$$

Proof. For each $i = 1, 2, \ldots, n$, set

$$g_i(\xi) = \begin{cases} a_i, & \xi \in (x_{i-1}, x_i), \\ 0, & \xi \in [a,b] \setminus (x_{i-1}, x_i), \end{cases}$$

and for each $i = 0, 1, 2, \ldots, n$, set

$$g_{x_i}(\xi) = \begin{cases} g(x_i), & \xi = x_i, \\ 0, & \xi \in [a,b] \setminus x_i. \end{cases}$$

Then

$$g = \sum_{i=1}^{n} g_i + \sum_{i=0}^{n} g_{x_i}.$$

Hence

$$\int_a^b A(f, dg) = \sum_{i=1}^{n} A(f(x_{i-1}) - f(x_i),\ a_i) + A(f(b), g(b)) - A(f(a), g(a))$$

$$= \sum_{i=1}^{n} \{A(f(x_{i-1}),\ a_i - g(x_{i-1})) + A(f(x_i),\ g(x_i) - a_i)\}$$

by the linearity of the integral over integrators and Corollaries 5 and 7. □

A series Σa_i in a normed linear space Z is said to be ***summable*** to a sum s if $s \in Z$ and the sequence of partial sums of the series converges to s. If this is the case, we write

$$s = \sum_i a_i .$$

The series Σa_i is said to be ***absolutely summable*** if $\Sigma \|a_i\| < \infty$.

Proposition 9 [10, p. 124]. *A normed linear space Z is a Banach space if and only if every absolutely summable series in Z is summable.*

Corollary 10. *Let $A \in L(X, Y; Z)$, and let $a = x_0 < x_1 < x_2 \ldots$ be a sequence of points in $[a,b]$ converging to b. Let g be a step function on $[a,b]$ with $g(\xi) = a_i$ when $x_{i-1} < \xi \le x_i$, for some vectors a_i in Y. If the series $\sum_i \|g(x_{i-1}+) - g(x_{i-1})\|$ is convergent, then for any bounded function $f : [a,b] \to X$, $\int_a^b A(f, dg)$ exists, and*

$$\int_a^b A(f, dg) = A(f(b), g(b) - g(b-)) + \sum_i A(f(x_{i-1}),\ a_i - g(x_{i-1})).$$

Proof. Let $\|f(x)\| \leq M$ for all $x \in [a,b]$. By assumption,

$$\sum \|A(f(x_{i-1}),\ a_i - g(x_{i-1}))\| = \sum \|A(f(x_{i-1}),\ g(x_{i-1}+) - g(x_{i-1}))\|$$

$$\leq M\|A\| \cdot \left\{ \sum_i \|g(x_{i-1}+) - g(x_{i-1})\| \right\} < \infty.$$

Since Z is a Banach space

$$\sum_i A(f(x_{i-1}),\ a_i - g(x_{i-1})) \in Z,$$

by Proposition 9.

Since $g(x_{i-1}+) = a_i$, using Theorem 8, we have

$$\int_a^{x_n} A(f,\ dg) = \sum_{i=1}^{n} A(f(x_{i-1}),\ a_i - g(x_{i-1})),$$

for each positive integer n.

Since $x_n \to b$, for every $x \in (a,b)$, there exists an integer n such that $x_{n-1} < x \leq x_n$. Thus, $\int_a^x A(f,\ dg)$ exists for all $x \in [a,b)$. Further,

$$\lim_{x \to b^-} \int_a^x A(f,\ dg) = \lim_{n \to \infty} \int_a^{x_n} A(f,\ dg) = \sum_i A(f(x_{i-1}),\ a_i - g(x_{i-1})).$$

Therefore $\int_a^b A(f,\ dg)$ exists, by Theorem 3, and

$$\int_a^b A(f,\ dg) = A(f(b),\ g(b) - g(b-)) + \sum_i A(f(x_{i-1}),\ a_i - g(x_{i-1})). \quad \square$$

References

[1] Canoy, S., Jr., *Bilinear Henstock-Stieltjes integrals in Banach spaces*, The Mindanao Forum **10** (1995) 63 - 76

[2] Canoy, S., Jr. and Jamil, F., *On some properties of a bilinear Henstock-Stieltjes integral*, Matimyas Matematika (to appear).

[3] Canoy, S., Jr. and Jamil, F., *On the strength of a bilinear Henstock-Stieltjes integral*, Matimyas Matematika (to appear).

[4] Canoy, S., Jr. and Jamil, F., *Integration by parts formula involving a bilinear transformation*, Proceedings of the Math and Physics International Conference (MSU-IIT) (to appear).

[5] Cao, S., *Henstock Integration in Banach Spaces*, Ph.D. Dissertation, U.P.-Diliman, 1991.

[6] Gordon, R., *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, American Mathematical Society, U.S.A., 1991.

[7] Jamil, F., *Henstock-Stieltjes Integrals in Banach Spaces*, Ph.D. Dissertation, MSU-IIT, 1999.

[8] Lee, P-Y., *Lanzhou Lectures on Henstock Integration*, World Scientific 1989.

[9] McLeod, R., *The Generalized Riemann Integral*, The Mathematical Association of America, U.S.A., 1989.

[10] Royden, H. L., *Real Analysis* 3e, Macmillan, N.Y., 1988.