The Mindanao Forum, Vol. XIV, No. 2 (December 1999)

Equivalence of Proof Techniques in Elementary Real Analysis

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Abstract

Several lemmas have been introduced by various authors to be used as tools in the proofs of theorems in real analysis. They come under various names: Cousin's Lemma, Thomson's Lemma, Creeping Lemma, Weak Creeping Lemma, Ford's Lemma, and Shanahan's Lemma. In this paper, they are shown to be equivalent.

There are many articles that introduce tools for proving theorems in real analysis. To cite some, we have Cousin's Lemma (see [3] and [6]), Thomson's Lemma (see [1] and [2]), the Creeping Lemma and the Weak Creeping Lemma (see [7]), Shanahan's Lemma (see [8]), and Ford's Lemma (see [5]). These lemmas have similar applications. For example, using any of these lemmas, we can prove the theorem: If f is continuous on $[a,b]$, then f is Riemann integrable on $[a,b]$. It is not surprising that these lemmas have similar applications since they are equivalent. It is the purpose of this paper to present a proof of the said equivalence.

Let us consider the lemmas and the definitions used.

A ***partition*** of an interval $[a,b]$ is a finite collection of non-overlapping closed intervals whose union is $[a,b]$. A ***tagged partition*** of $[a,b]$ is a partition with one point, referred to as a ***tag***, chosen from each sub-interval comprising the partition. A tagged partition of $[a,b]$ will be denoted by $\{(c_i;[x_{i-1},x_i]) : 1 \le i \le n\}$ where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and $c_i \in [x_{i-1}, x_i]$ is the tag of the interval $[x_{i-1},x_i]$ for each i. Now let δ be a positive function defined on $[a,b]$. A ***δ-fine tagged partition*** of $[a,b]$ is a tagged partition $\{(c_i;[x_{i-1}, x_i]) : 1 \le i \le n\}$ of $[a,b]$ that satisfies $[x_{i-1},x_i] \subseteq (c_i - \delta(c_i), c_i + \delta(c_i))$ for each i.

Cousin's Lemma (C_1L). *If* δ is *a positive function defined on the inter-val* $[a,b]$, *then there exists a* δ*-fine tagged partition of* $[a,b]$.

For a proof of this, see [3] or [6].

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A collection ζ of closed subintervals of $[a,b]$ is a ***full cover*** of $[a,b]$ if for each $x \in [a,b]$ there corresponds a number $\delta(x) > 0$ such that every closed subinterval of $[a,b]$ that contains x and has length less than $\delta(x)$ belongs to ζ.

Thomson's Lemma (TL). *If ζ is a full cover of $[a,b]$, then ζ contains a partition of $[a,b]$.*

For a proof of this, see [1] or [2].

Creeping Lemma (C_2L). *Let ρ be a transitive relation on the interval $[a,b]$. If each $x \in [a,b]$ has a neighborhood N_x such that $u\rho v$ whenever $u \in [a,x] \cap N_x$ and $v \in [x,b] \cap N_x$, $u < v$, then $a\rho b$.*

For a proof of this see [7].

Weak Creeping Lemma (WCL). *Let ρ be a transitive relation on the interval $[a,b]$. If each $x \in [a,b]$ has a neighborhood N_x such that $u\rho v$ whenever $u, v \in [a,b] \cap N_x$, $u < v$, then $a\rho b$.*

The proof of this lemma follows directly from the Creeping Lemma.

A statement $P(I)$ concerning intervals I will be called ***interval-additive*** if whenever $P(I_1)$ and $P(I_2)$ are true and $\text{int}(I_1 \cap I_2) \neq \varnothing$ then $P(I_1 \cup I_2)$ is also true.

We say that a proposition P is ***true at a point*** $x_0 \in [a,b]$ if $P(I_0)$ is true for some subinterval I_0 of $[a,b]$ such that $x_0 \in \text{int } I_0$ (with respect to the relative topology on $[a,b]$).

Ford's Lemma (FL). *If P is an interval-additive proposition that is true at each point of $[a,b]$, then $P([a,b])$ is true.*

For a proof of this, see [5].

Let ζ be a family of subsets of $[a,b]$. Let us say that ζ is ***local*** if each $x \in [a,b]$ has a neighborhood, with respect to the relative topology on $[a,b]$, which is a member of ζ. We say that ζ is ***additive*** if whenever $C_1, C_2 \in \zeta$ such that $C_1 \cap C_2 \neq \varnothing$, then $C_1 \cup C_2 \in \zeta$.

Shanahan's Lemma (SL). *If ζ is local, additive family of closed subintervals of $[a,b]$, then $[a,b] \in \zeta$.*

For the proof of this, see [8].

We will now prove that these lemmas are equivalent. For the direction of the proof, we consider the following :

$$C_1L \Rightarrow TL \Rightarrow C_2L \Rightarrow WCL \Rightarrow FL \Rightarrow SL \Rightarrow C_1L$$

Proposition 1. *Cousin's Lemma implies Thomson's Lemma.*

Proof. Let ζ be a full cover of $[a,b]$. Then for each $x \in [a,b]$, there corresponds a number $\delta_1(x) > 0$ such that every closed subinterval of $[a,b]$ that contains x and has length less than $\delta_1(x)$ belongs to ζ. Define

$$\delta_2(x) = \frac{1}{2}\delta_1(x), \; x \in [a,b].$$

Clearly, $\delta_2(x) > 0$, for all $x \in [a,b]$. By Cousin's Lemma, there exists a δ_2-fine tagged partition $\{(c_i;[x_{i-1},x_i]) : 1 \leq i \leq n\}$ of $[a,b]$. Since

$$c_i \in [x_{i-1},x_i] \subseteq (c_i - \delta_2(c_i), c_i + \delta_2(c_i))$$

for each i, then each $[x_{i-1},x_i]$ contains c_i and its length $x_i - x_{i-1} < \delta_1(c_i)$. Thus, $[x_{i-1},x_i] \in \zeta$ for each i. Hence, ζ contains a partition of $[a,b]$. □

Proposition 2. *Thomson's Lemma implies the Creeping Lemma.*

Proof. Let ρ be a relation satisfying the hypothesis of the Creeping Lemma. Let $x \in [a,b]$. Then, there exists a neighborhood N_x of x such that $u\rho v$ whenever $u \in N_x \cap [a,x]$ and $v \in N_x \cap [x,b]$, $u < v$. It follows then that there is an open interval I_x such that $x \in I_x \subseteq N_x$. Define ζ_x as follows:

$$\zeta_x = \{[u,v] : u \in I_x \cap [a,x] \text{ and } v \in I_x \cap [x,b]\}.$$

Let $m_x = \inf\{u : u \in I_x \cap [a,x]\}$ and $M_x = \sup\{v : v \in I_x \cap [x,b]\}$. Let $\zeta = \cup\{\zeta_x : x \in [a,b]\}$. We note the following obvious remarks:

1. ζ is a collection of closed subintervals of $[a,b]$;
2. m_x and M_x exist for any $x \in [a,b]$;
3. $m_a = a < M_a$ and $m_b < M_b = b$;
5. $m_x < x < M_x$ for each $x \in (a,b)$.

If $[u,v] \in \zeta$, then $u\rho v$.

Claim: ζ is a full cover of $[a,b]$.

$$\text{Let } \delta(x) = \begin{cases} \min\{x - m_x, M_x - x\}, & \text{if } x \in (a,b), \\ \max\{x - m_x, M_x - x\}, & \text{if } x = a \text{ or } x = b. \end{cases}$$

Clearly, $\delta(x) > 0$, for all $x \in [a,b]$. Let $[u,v]$ be a subinterval of $[a,b]$ that contains x and has length less than $\delta(x)$. Consider the following cases:

(*i*) $x = a$. Then $x = a = u$. Clearly, $v - a < \delta(a) = M_a - a$, thus $u = a < v < M_a$. It follows then that $[u,v] \in \zeta_a \subseteq \zeta$.

(*ii*) $x = b$. Then $x = b = v$. Clearly, $b - u < \delta(b) = b - m_b$, thus $m_b < u < v = b$. It follows that $[u,v] \in \zeta_b \subseteq \zeta$

(*iii*) $x \in (a,b)$. Then, $x - u \leq v - u < \delta(x) \leq x - m_x$ and $v - x \leq v - u < \delta(x) \leq M_x - x$. Thus, $m_x < u \leq x \leq v < M_x$. In effect, $u \in I_x \cap [a,x]$ and $v \in I_x \cap [x,b]$. Hence, $[u,v] \in \zeta_x \subseteq \zeta$. Therefore, ζ is a full cover.

By Thomson's Lemma, ζ contains a partition $\{[x_{i-1}, x_i] : 1 \leq i \leq n\}$ of $[a,b]$. Since $[x_{i-1}, x_i] \in \zeta$, then $x_{i-1} \rho x_i$ for each i. Hence, by transitivity, $a \rho b$. □

Proposition 3. *The Creeping Lemma implies the Weak Creeping Lemma.*

The proof follows directly.

Proposition 4. *The Weak Creeping Lemma implies Ford's Lemma.*

Proof. Let P be a proposition satisfying the hypothesis of Ford's Lemma. Define a relation ρ on $[a,b]$ as follows: For $u, v \in [a,b]$, $u \rho v$ if and only if there exists a subinterval I of $[a,b]$ such that $u, v \in \text{int } I$ (interior of I with respect to the relative topology on $[a,b]$) and $P(I)$ is true.

Suppose $u \rho v$ and $v \rho w$. Then there exist subintervals I_1 and I_2 of $[a,b]$ such that $u, v \in \text{int } I_1$, $v, w \in \text{int } I_2$, and $P(I_1)$ and $P(I_2)$ are true. Since $I_1 \cap I_2 \neq \varnothing$, then $I_1 \cup I_2$ is a subinterval of $[a,b]$. Also, $u, w \in \text{int } I_1 \cup \text{int } I_2 \subseteq \text{int}(I_1 \cup I_2)$. Moreover, $\text{int}(I_1 \cap I_2) \neq \varnothing$, thus $P(I_1 \cup I_2)$ is true. Hence, $u \rho w$. This shows that ρ is transitive.

Let $x \in [a,b]$. Then there exists a subinterval I_x of $[a,b]$ such that $P(I_x)$ is true and $x \in \text{int } I_x$ with respect to the relative topology. Let $N_x = \text{int } I_x$. Let $u, v \in N_x \cap [a,b]$, $u < v$. Then, $u, v \in \text{int } I_x$. Now, $P(I_x)$ is true, thus $u \rho v$.

Hence, by the Weak Creeping Lemma, $a \rho b$, i.e., there exists a subinterval I of $[a,b]$ such that $a, b \in \text{int}(I)$ with respect to the relative topology and $P(I)$ is true. Clearly, $I = [a,b]$. Hence, $P([a,b])$ is true. □

Proposition 5. *Ford's Lemma implies Shanahan's Lemma.*

Proof. Let ζ be local, additive family of closed subintervals of $[a,b]$. Let $P(I)$ be the proposition "$I \in \zeta$".

Suppose $P(I_1)$ and $P(I_2)$ are true where $\text{int}(I_1 \cap I_2) \neq \emptyset$. Then $I_1, I_2 \in \zeta$ and $I_1 \cap I_2 \neq \emptyset$. Since ζ is additive then $I_1 \cup I_2 \in \zeta$. Thus, $P(I_1 \cup I_2)$ is true. This shows that P is interval-additive.

Let $x \in [a,b]$. Since ζ is local then there exists a neighborhood N_x of x such that $N_x \in \zeta$. This implies that N_x is a closed subinterval of $[a,b]$ and $P(N_x)$ is true. Clearly, $x \in \text{int}\, N_x$. Thus P is true at x.

Hence, by Ford's Lemma, $P([a,b])$ is true, i.e., $[a,b] \in \zeta$. □

Proposition 6. *Shanahan's Lemma implies Cousin's Lemma.*

Proof. In [8], the Heine-Borel Theorem is shown equivalent to Shanahan's Lemma while in [4], it is shown equivalent to Cousin's Lemma. Thus, the implica-tion follows (actually equivalent). □

To this end, it is worth noting that these lemmas are equivalent to the following important theorems:

1. The Least Upper Bound Property
2. The Heine-Borel Theorem
3. The Bolzano-Weierstrass Theorem
4. The Monotone Sequence Property
5. The Cauchy Convergence Criterion
6. The Nested Sets Property

The equivalence of these theorems together with Cousin's Lemma and Thomson's Lemma are shown in [4].

The author would like to extend his gratitude to Dr. Carpio for reading this paper.

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