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On Connected Graphs G Such That $L^3(G)$ is Eulerian But $L^2(G)$ is Not

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Abstract

A simple graph G is said to possess property π if $L^3(G)$ is Eulerian but $L^2(G)$ is not. If G is a connected simple graph satisfying π, then G is a path of order 4. This result is useful in constructing disconnected simple graphs satisfying property π.

Let G be a simple graph with nonempty edge-set. The ***line graph*** of G, denoted by $L(G)$, is the graph whose vertices are the edges of G and where two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are. The following remark is immediate from definition.

Remark 1.0. *If $u = x{\cdot}y$ is an edge of G, then*

$$deg_{L(G)}\, u = deg_G\, x + deg_G\, y - 2.$$

A graph is ***eulerian*** if it has a spanning closed walk that traverses each edge exactly once. We say that a graph H possesses property π if $L^3(H)$ is eulerian but $L^2(H)$ is not. Our goal in this paper is to find all connected simple graphs with property π. We will also give a way of constructing disconnected graphs satisfying the same property.

1. Preliminaries

Lemma 1.1. (1) $L(P_n) \cong P_{n-1}$ *for any path* P_n, $n \geq 2$.

(2) $L(C_n) \cong C_n$ *for all cycle* C_n.

Proof. See [G]. □

Lemma 1.2. *A connected graph G is eulerian if and only if every vertex of G has even degree.*

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Proof. See [H]. □

Corollary 1.2.1. *The following statements are equivalent for a nontrivial connected graph G:*

(1) *$L(G)$ is eulerian.*

(2) *The degrees of all the vertices of G have the same parity.*

Proof. (1) ⇒ (2). Let x be a vertex of G. Since G is a nontrivial connected graph, there is another vertex y of G such that xy is an edge of G. Now, $L(G)$ is eulerian. By Remark 1.0 and Lemma 1.2, $\deg_G x + \deg_G y$ is even. This means that the degrees of x and the vertices adjacent to x have the same parity. Since G is connected, it follows that all the vertices of G have the same parity.

(2) ⇒ (1). Let u be a vertex of $L(G)$. Then $u = xy$ for some edge xy of G. Since $\deg_G x$ and $\deg_G y$ have the same parity, it follows from Remark 1.0 that $\deg_G u$ is even. Thus, every vertex of $L(G)$ has even degree. By Lemma 1.2, $L(G)$ is eulerian. □

Lemma 1.3. *If G is a connected graph with $|V(G)| \geq 2$, then $L(G)$ is connected.*

Proof. Clearly, $L(G)$ is connected if $|V(G)| = 2$. Suppose now that $|V(G)| \geq 3$. Then G has at least two edges. Now, let u and v be distinct vertices of $L(G)$. Then there exist distinct edges ab and cd of G such that $u = ab$ and $v = cd$. Since G is connected, there is a walk $x_1x_2...x_k$ in G, where $x_1 = b$ and $x_k = c$. This implies that $ax_1x_2...x_kd$ is a walk in G joining a and d. Let $w_j = x_jx_{j+1}$ for $j = 1, 2, ..., k-1$. Then $uw_1w_2...w_{k-1}v$ is a walk in $L(G)$ joining u and v. Therefore, $L(G)$ is connected. □

2. Connected Graphs With Property π

Let G be a connected graph which satisfies property π. Then $L(G)$, $L^2(G)$ and $L^3(G)$ are all connected by Lemma 1.3. Since $L^3(G)$ is eulerian, it follows from Corollary 1.2.1 that the degrees of all the vertices of $L^2(G)$ have the same parity. But $L^2(G)$ is not eulerian; hence, by Lemma 1.2, the degrees of the vertices of $L^2(G)$ have odd parity. This means that $\deg_{L(G)} u + \deg_{L(G)} v$ is odd for every edge uv of $L(G)$. Thus, the degrees of adjacent vertices in $L(G)$ have opposite parity. Consequently, $L(G)$ does not have an odd cycle. Observe that no vertex of G can have a degree greater than or equal to three; otherwise, $L(G)$ would contain a complete subgraph K_3 which is an odd cycle. It is now clear that any spanning tree of G must be a path. Hence, G is either a path or a cycle. Note that $|V(G)| \geq 3$ since G possesses property π. By Lemma 1.1, $L^2(G)$ is a path or a cycle. But $L^2(G)$ is not eulerian; hence G is not a cycle. Thus $L^2(G)$ is a path, and so $L^3(G)$ is also a path by Lemma 1.1. But $L^3(G)$ is eulerian; hence $L^3(G)$ must be a complete graph K_1. Therefore, G is isomorphic to a path P_4. We summarize our result as

Theorem 2.4. *If G is a connected graph such that $L^2(G)$ is not eulerian but $L^3(G)$ is, then G is a path of order* 4.

Remarks for Disconnected Graphs With Property π

From Theorem 2.4 we see that there is only one connected graph (up to isomorphism), which satisfies property π. This, however, is not true for disconnected graphs. The graphs $P_4 \dot{\cup} P_k$ for $k = 1, 2, 3$ and $C_n \dot{\cup} P_3$ are disconnected, but they satisfy property π. It is recommended to the reader to find all disconnected simple graphs possessing property π.

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