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The (r,β)-Stirling Numbers

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1. Introduction

The (r,β)-***Stirling numbers***, denoted by

$$\left\langle {n \atop k} \right\rangle_{\beta,r},$$

extend the concept of r-Stirling numbers of the second kind by introducing a new parameter β. We define the (r,β)-Stirling numbers by means of the linear transformation

$$t^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} (t-r)_{\beta,k}, \qquad (1)$$

where

$$(t-r)_{\beta,k} = \prod_{i=0}^{k-1} (t-r-i\beta).$$

We call $(t)_{\beta,k}$ the ***generalized factorial of*** t ***with increment*** β. In particular, for $\beta = 1$ and $r = 0$, equation (1) will reduce to

$$t^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{1,0} (t)_k,$$

which yields the ***ordinary Stirling numbers of the second kind*** $S(n,k)$ given in [2]. Here we have

$$S(n,k) = \left\langle {n \atop k} \right\rangle_{1,0}.$$

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In fact, the r-Stirling numbers of the second kind can be written as

$$\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{1,r}.$$

This will be shown in later. Furthermore, if $\beta = 0$ and $r = 1$, we get

$$\left(\begin{matrix} n \\ k \end{matrix} \right) = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{0,1}.$$

2. Generating Functions

In this section, we will consider two types of generating function, viz. the exponential generating function and the rational generating function. But first let us mention an explicit formula for the (r,β)-Stirling numbers.

Theorem 1. *The (r,β)-Stirling numbers satisfy the following explicit formula*

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\beta,r} = \frac{1}{\beta^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (\beta j + r)^n. \tag{2}$$

Proof. Note that we can rewrite (1) as follows

$$t^n = \sum_{k=0}^{n} \binom{\frac{t-r}{\beta}}{k} \beta^k k! \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\beta,r}.$$

Replacing t with $\beta t + r$, we have

$$(\beta t + r)^n = \sum_{k=0}^{n} \binom{t}{k} \beta^k k! \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\beta,r}.$$

Using Newton's Interpolation Formula, with

$$f(t) = (\beta t + r)^n \text{ and } \Delta^k f(0) = \beta^k k! \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{\beta,r},$$

we get

$$\beta^k k!\left\langle{n \atop k}\right\rangle_{\beta,r} = [\Delta^k(\beta t + r)^n]_{t=0} = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j + r)^n.$$

This is precisely equivalent to (2). □

Clearly, (2) does not work when $\beta = 0$. However, we can let β approach zero to get a suitable limit. This is possible because, from (2),

$$\lim_{\beta\to 0}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j + r)^n = r^n\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} = 0$$

Hence, the limit of (2) as $\beta \to 0$ is an indeterminate form. After k applications of l'Hospital's rule, we get

$$\lim_{\beta\to 0}\left\langle{n \atop k}\right\rangle_{\beta,r} = \binom{n}{k}r^{n-k}. \tag{3}$$

We are now ready to mention the exponential generating function.

Theorem 2. *For $\beta \neq 0$, we have*

$$\sum_{n\geq 0}\left\langle{n \atop k}\right\rangle_{\beta,r}\frac{t^n}{n!} = \frac{1}{\beta^k k!}e^{rt}(e^{\beta t} - 1)^k. \tag{4}$$

Proof: Making use of Theorem 1, we have

$$\sum_{n\geq 0}\left\langle{n \atop k}\right\rangle_{\beta,r}\frac{t^n}{n!} = \frac{1}{\beta^k k!}\sum_{n\geq 0}\left\{\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j + r)^n\right\}\frac{t^n}{n!}$$

$$= \frac{1}{\beta^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left\{\sum_{n\geq 0}(\beta j + r)^n\frac{t^n}{n!}\right\}$$

$$= \frac{1}{\beta^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left\{\sum_{n\geq 0}\left[\sum_{i=0}^{n}\binom{n}{i}(\beta j)^{n-i}r^i\right]\frac{t^n}{n!}\right\}$$

$$= \frac{1}{\beta^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left\{ \sum_{n \geq 0} \left[\sum_{i=0}^{n} \frac{(rt)^i}{i!} \frac{(\beta j t)^{n-i}}{(n-i)!} \right] \right\}.$$

By Cauchy's formula for the product of two power series [4], we get

$$\sum_{n \geq 0} \left\langle {n \atop k} \right\rangle_{\beta,r} \frac{t^n}{n!} = \frac{1}{\beta^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left\{ \sum_{\lambda \geq 0} \frac{(rt)^\lambda}{\lambda!} \sum_{\mu \geq 0} \frac{(\beta j t)^{n-\mu}}{(n-\mu)!} \right\}.$$

Thus,

$$\sum_{n \geq 0} \left\langle {n \atop k} \right\rangle_{\beta,r} \frac{t^n}{n!} = \frac{1}{\beta^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left\{ e^{rt} e^{\beta j t} \right\}$$

$$= \frac{1}{\beta^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (e^{\beta t})^j.$$

Using Binomial Theorem, we obtain (4). □

If $\beta = 1$, (4) gives

$$\sum_{n \geq 0} \left\langle {n \atop k} \right\rangle_{\beta,r} \frac{t^n}{n!} = \frac{1}{k!} e^{rt} (e^t - 1)^k,$$

which implies that (see [1, eq. (38)])

$$\left\{ {n+r \atop k+r} \right\}_r = \left\langle {n \atop k} \right\rangle_{1,r}.$$

The following theorem states the rational generating function for (r,β)-Stirling numbers.

Theorem 3. *The (r,β)-Stirling numbers have the rational generating function*

$$\phi_k(\beta, r) = \sum_{n \geq 0} \left\langle {n \atop k} \right\rangle_{\beta,r} t^n = \frac{t^k}{\prod_{j=0}^{k} [1 - (\beta j + r)t]}. \qquad (5)$$

Proof: Consider the equation

$$\frac{1}{\prod_{j=0}^{k}[1-(\beta j+r)t]}=\sum_{j=0}^{k}\frac{A_j}{1-(\beta j+r)t}. \tag{6}$$

Using the method of partial fractions, we can rewrite (6) as follows

$$\sum_{j=0}^{k}\left[A_j\prod_{j_1=0}^{j-1}[1-(\beta j_1+r)t]\prod_{j_2=j+1}^{k}[1-(\beta j_2+r)t]\right]=1.$$

Taking $t=(\beta j+r)^{-1}$, we have

$$A_j=\frac{1}{\beta^k k!}(-1)^{k-j}(\beta j+r)^k,\quad j=0,1,2,\ldots,k.$$

Substitution to equation (6) gives

$$\frac{t^k}{\prod_{j=0}^{k}[1-(\beta j+r)t]}=\frac{1}{\beta^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{(\beta j+r)^k}{1-(\beta j+r)t}$$

$$=\sum_{j=0}^{k}\frac{(-1)^{k-j}\binom{k}{j}(\beta j+r)^k}{\beta^k k!}\left\{\sum_{\nu\geq 0}[(\beta j+r)t]^{\nu}\right\}$$

$$=\sum_{\nu\geq 0}\left[\frac{1}{\beta^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j+r)^{k+\nu}\right]t^{k+\nu}.$$

Replacing $k+\nu$ with n, we obtain

$$\frac{t^k}{\prod_{j=0}^{k}[1-(\beta j+r)t]}=\sum_{n\geq k}\left[\frac{1}{\beta^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j+r)^n\right]t^n$$

$$= \sum_{n\geq k} \left\langle {n \atop k} \right\rangle_{\beta,r} t^n . \qquad \square$$

If $\beta = 1$, equation (5) gives Corollary 10 in [1]. Moreover, if $\beta = 1$ and $r = 0$, equation (5) will reduce to the rational generating function of the second kind of ordinary Stirling numbers (see [2],[4]).

Note that we can rewrite equation (5) as follows

$$\sum_{n\geq k} \left\langle {n \atop k} \right\rangle_{\beta,r} t^{n-k} = \prod_{j=0}^{k} \left[\sum_{c_j \geq 0} (\beta j + r)^{c_j} t^{c_j} \right]$$

$$= \sum_{c_0+c_1+\cdots+c_k \geq 0} \left[\prod_{j=0}^{k} (\beta j + r)^{c_j} \right] t^{c_0+c_1+\cdots+c_k}$$

$$= \sum_{n\geq k} \left[\sum_{c_0+c_1+\cdots+c_k = n-k} \left\{ \prod_{j=0}^{k} (\beta j + r)^{c_j} \right\} \right] t^{n-k} .$$

Identifying the coefficients of the term t^{n-k}, we obtain another explicit formula for (r,β)-Stirling numbers.

Theorem 4. *The following explicit formula holds*

$$\left\langle {n \atop k} \right\rangle_{\beta,r} = \sum_{c_1+c_2+\cdots+c_k = n-k} \left[\prod_{j=0}^{k} (\beta j + r)^{c_j} \right] . \quad (7)$$

Note that this formula will work whatever the value of β is. In fact, if $\beta = 0$, equation (7) gives

$$\left\langle {n \atop k} \right\rangle_{0,r} = \sum_{c_1+c_2+\cdots+c_k = n-k} \left[\prod_{j=0}^{k} r^{c_j} \right] = \binom{n}{k} r^{n-k},$$

which is precisely equivalent to (3).

3. Recurrence Relations

Recurrence relations are useful tool in computing the first values of (r,β)-Stirling numbers. In this section, we will consider three types of recurrence relations; the triangular, horizontal, and vertical recurrence relations. The next theorem provides the first type of these recurrence relations.

Theorem 5. *The (r,β)-Stirling numbers satisfy the following 'triangular' recurrence relation*

$$\left\langle {n+1 \atop k} \right\rangle_{\beta,r} = \left\langle {n \atop k-1} \right\rangle_{\beta,r} + (k\beta + r)\left\langle {n \atop k} \right\rangle_{\beta,r}. \tag{8}$$

Proof: Using relation (1), we have

$$\sum_{k=0}^{n+1} \left\langle {n+1 \atop k} \right\rangle_{\beta,r} (t-r)_{\beta,k} = t^n[t - r - k\beta + (k\beta + r)] =$$

$$= \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} (t-r)_{\beta,k}(t - r - k\beta) + \sum_{k=0}^{n} (k\beta + r)\left\langle {n \atop k} \right\rangle_{\beta,r} (t-r)_{\beta,k}$$

$$= \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} (t-r)_{\beta,k+1} + \sum_{k=0}^{n} (k\beta + r)\left\langle {n \atop k} \right\rangle_{\beta,r} (t-r)_{\beta,k}$$

Comparing the coefficients of the term $(t-r)_{\beta,k}$, we obtain (8). □

We know that

$$\left\{ {n+r \atop k+r} \right\}_r = \left\langle {n \atop k} \right\rangle_{1,r}.$$

Now, taking $\beta = 1$, equation (8) gives

$$\left\langle {n+1 \atop k} \right\rangle_{1,r} = \left\langle {n \atop k-1} \right\rangle_{1,r} + (k+r)\left\langle {n \atop k} \right\rangle_{1,r}.$$

Replacing n with $n-r$ and k with $k-r$, we have

$$\left\{ {n+1 \atop k} \right\}_r = \left\{ {n \atop k-1} \right\}_r + k\left\{ {n \atop k} \right\}_r.$$

This is the triangular recurrence relation given in [1, eq. 8]. The next theorem gives the second type of recurrence relations.

Theorem 6. *The* (r,β)*-Stirling numbers satisfy the following horizontal recurrence relation*

$$\left\langle {n \atop k} \right\rangle_{\beta,r} = \sum_{j=0}^{n-k} (-1)^j ((k+1)\beta + r)_{-\beta,j} \left\langle {n+1 \atop k+j+1} \right\rangle_{\beta,r}. \qquad (9)$$

Proof. Using Theorem 5,

$$\text{the RHS of (9)} = \sum_{j=0}^{n-k} (-1)^j ((k+1)\beta + r)_{-\beta,j} \left\langle {n \atop k+j} \right\rangle_{\beta,r} +$$

$$+ \sum_{j=0}^{n-k} (-1)^j ((k+1)\beta + r)_{-\beta,j+1} \left\langle {n \atop k+j+1} \right\rangle_{\beta,r}$$

$$= \sum_{j=0}^{n-k} (-1)^{j+1} ((k+1)\beta - n\alpha + r)_{-\beta,j+1} \left\langle {n \atop k+j+1} \right\rangle_{\beta,r} +$$

$$+ \sum_{j=0}^{n-k} (-1)^j ((k+1)\beta - n\alpha + r)_{-\beta,j+1} \left\langle {n \atop k+j+1} \right\rangle_{\beta,r}$$

$$= \left\langle {n \atop k} \right\rangle_{\beta,r}. \quad \square$$

The next theorem gives the vertical recurrence relation for (r,β)-Stirling numbers.

Theorem 7. *The* (r,β)*-Stirling numbers satisfy the following vertical recurrence relation*

$$\left\langle {n+1 \atop k+1} \right\rangle_{\beta,r} = \sum_{j=k}^{n} \left\langle {j \atop k} \right\rangle_{\beta,r} ((k+1)\beta + r)^{n-j}. \qquad (10)$$

Proof: From equation (5), we get

$$\sum_{n \geq k} \left\langle {j \atop k} \right\rangle_{\beta,r} t^n = t(1 - (\beta k + r)t)^{-1} \phi_{k-1}(\beta, r)$$

$$= \sum_{j \geq k} \sum_{m \geq 0} \left\langle {j-1 \atop k-1} \right\rangle_{\beta,r} (\beta k + r)^m t^{j+m}$$

Replacing $j+m$ with n, we have

$$\sum_{n \geq k} \left\langle {n \atop k} \right\rangle_{\beta,r} t^n = \sum_{j \geq k} \sum_{n \geq j} \left\langle {j-1 \atop k-1} \right\rangle_{\beta,r} (\beta k + r)^{n-j} t^n$$

$$= \sum_{n \geq k} \sum_{k \leq j \leq n} \left\langle {j-1 \atop k-1} \right\rangle_{\beta,r} (\beta k + r)^{n-j} t^n.$$

Comparing the coefficients of the term t^n, we obtain

$$\left\langle {n \atop k} \right\rangle_{\beta,r} = \sum_{j=k}^{n} \left\langle {j-1 \atop k-1} \right\rangle_{\beta,r} (\beta k + r)^{n-j}. \quad \square$$

Taking $\beta = 0$ and $r = 1$, Theorem 7 will give

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k},$$

which is known to be ***Chu-Shih-Chieh's identity***.

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