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A Construction of Bernştein Sets

CALIXTO B. FABRERO, JR.

Abstract

In this paper, we construct an example of a Lebesgue nonmeasurable set of real numbers using their topological properties. Such a construction is due to F. Bernstein.

Key Words: Bernstein set, Lebesgue-Stieltjes measure

One well-known example of a nonmeasurable set, whose construction was due to Vitali, is found in many books of real variable theory and measure theory (e.g. see [1, Theorem 10.28]). Such a construction is based on the algebraic properties of the set **R** of real numbers.

In this paper, we construct a subset B of real numbers that is measurable for no measure λ_α with continuous α. The measure λ_α is the Lebesgue-Stieltjes measure corresponding to a nondecreasing function α on **R**. Note that if $\alpha(x) = x$, the measure λ_α is the Lebesgue measure on **R**.

If Λ_α is the σ - algebra of all λ_α - measurable subsets of **R**, we will show that $\Lambda_\alpha \neq P(\mathbf{R})$.

Throughout this paper, c denotes the cardinality of the continuum and $\mathfrak{F}$ is the family of all uncountable closed subsets of **R**. The Principle of Transfinite Induction or Recursion, the Axiom of Choice, and some basic facts about set theory (see [2, Chapters I and II]) will be assumed.

Recall that the measure λ_α is ***inner-regular*** if for each $A \in \Lambda_\alpha$

$$\lambda_\alpha(A) = \sup \{ \lambda_\alpha(K) : K \text{ is compact}, K \subset A \}.$$

The measure λ_α is ***outer-regular*** if for each $A \in \Lambda_\alpha$,

$$\lambda_\alpha(A) = \inf \{ \lambda_\alpha(V) : V \text{ is open}, A \subset V \}.$$

We shall need the following theorems:

✍ CALIXTO B. FABRERO, JR. teaches physics at MSU-Iligan Institute of Technology. His main interest is in analysis, particularly, measure theory and topology. He has BS degrees in physics and mathematics from the MSU and an MS in mathematics from MSU-IIT. In addition to teaching, he also enjoys playing competitive table tennis and simply being alive.

1 Theorem. *Let* $\boldsymbol{\alpha}$ *be a cardinal number. Then there exists an ordinal number* α *such that* $P_\alpha = \boldsymbol{\alpha}$.

Proof. (See [1], p. 29.)

The set P_α is the set of all ordinals $< \alpha$. If $(W, \leq)$ is a well-ordered set, the set $I(a) = \{ x \in W : x \leq a, x \neq a \}$ is called ***the initial segment of W determined by** a*.

2 Theorem (The Principle of Transfinite Induction). *Let* $(W, \leq)$ *be a well-ordered set and let* $A \subset W$ *be such that* $a \in A$ *whenever* $I(a) \subset A$. *Then* $A = W$.

Proof. See [1, p. 17].

3 Theorem. *Let* U *be a nonempty open subset of* $\mathbf{R}$. *Then there exists one and only one pairwise disjoint family* I *of open intervals of* $\mathbf{R}$ *such that* $U = \cup$I. *The family* I *is countable and the members of* I *are called the component intervals of* U. *For each* $I \in$ I, *the end points of* I *are not in* U.

Proof. See [1, p. 69].

4 Theorem. *Let* X *be a complete metric space and let* A *be a nonempty perfect subset of* X. *Then* $|A| \geq \mathbf{c}$..

Proof. See [1, p. 72].

A set A is said to be ***perfect*** if it is closed and has no isolated points, i.e., if A is equal to the set of its own limit points.

5 Theorem. (Cantor-Bendixson). *Let* X *be a topological space with a countable base* B *and let* A *be any closed subset of* X. *Then* X *contains a perfect subset* P *and a countable subset* C *such that* $A = P \cup C$.

Proof. See [1, pp. 72-73].

6 Theorem. *Let* X *be a locally compact Hausdorff space and let* λ_α *be the Lebesgue-Stieltjes measure corresponding to a function* α. *Let* A *be a* λ_α *-measurable subset of* X *such that* $A \subset \bigcup_{n=1}^{\infty} B_n$ *for some sequence* $\{B_n\}$ *of subsets of* X *with* $\lambda_\alpha(B_n) < \infty$ *for all* n. *Then*

$$\lambda_\alpha(A) = \sup \{\lambda_\alpha(F) : F \text{ is compact}, F \subset A\}.$$

Proof. See [1, pp. 137-138].

7 Remark. Every Euclidean space $\mathbf{R}^n$ satisfies the hypotheses of Theorems 4, 5 and 6.

8 Proposition. *Every uncountable closed subset F of* **R** *has a cardinal number c.*

Proof. Let $\mathfrak{I}$ be the family of all uncountable closed subsets of **R**. Then by Theorem 5, for each $F \in \mathfrak{I}$, we have $F = P \cup C$, where P is perfect and C is countable. Since $F \neq \varnothing$, we have $P \neq \varnothing$ and thus by Theorem 4, $|P| \geq c$. Hence, $|F| = |P \cup C| \geq |P| \geq c$. Since F is uncountable and $|F| \leq |\mathbf{R}| = c$, we must have $|F| = c$ for each $F \in \mathfrak{I}$. □

9 The Construction. We now proceed with the construction of a subset B of **R** such that $B \cap F \neq \varnothing$ and $(\mathbf{R} - B) \cap F \neq \varnothing$ for every $F \in \mathfrak{I}$. Such a set is called a ***Bernstein set***, named after F. Bernstein. Denote by $I(x,r)$ the open interval with center x and radius $r > 0$. Let B = $\{ I(\xi_n, r_n) : \xi_n \in \mathbf{Q}, r_n \in \mathbf{Q}^+ \}$, for $n \geq 1$. The family B is a countable basis for **R** (See [2, Example 4, p. 65]), so, $|\mathrm{B}| = \aleph_0$. Let G be the class of all open subsets of **R** and let $G \in$ G. Then there exists a family I $\subset$ B such that G $= \cup$I. This is not a disjoint union, however, by Theorem 3, there exists a unique countable pairwise disjoint family I* of open intervals in **R** such that G $= \cup$I*. By the uniqueness of I*, it suffices to use a base B′ $\subset$ B such that the elements of B′ are pairwise disjoint.

Let f : P(B′) $\to$ G be defined by f(I) $= G$ for each $G \in$ G. Thus, by Theorem 3, there exists a unique pairwise disjoint countable family I $\subset$ B′ such that $G = \cup\{ I : I \in \mathrm{I}\}$. Thus f is onto. By Theorem 3 again, if G, H are in G and $G = H$, we can choose a unique pairwise disjoint countable families I and V in P(B′) such that

$$G = \cup\{ I : I \in \mathrm{I} \} \text{ and } H = \cup\{ V : V \in \mathrm{V} \}.$$

Hence,

$$\cup\{ I : I \in \mathrm{I} \} = \cup\{ V : V \in \mathrm{V} \}$$

and by the uniqueness of these representations we must have I = V. Therefore, f is one-to-one and |G| = |P(B′)| = $2^{\aleph_0} = c$. By complementation, there are at most c closed subsets of **R**. Since there are c many closed intervals, there are at least c uncountable closed subsets of **R**, i.e., $|\mathfrak{I}| = c$.

By Theorem 1, for the cardinal number c, we can choose an ordinal number ω_c such that $|P_{\omega_c}| = c$, where $P_{\omega_c} = \{\alpha : \alpha < \omega_c\}$. Again by Theorem 1, P_{ω_c} is a well-ordered set and ord $P_{\omega_c} = \omega_c$. Thus, ω_c has c predecessors and it is the smallest ordinal number of cardinality c. By the Well-Ordering Theorem and Proposition 8, $\mathfrak{I}$ can be indexed by ordinals $< \omega_c$, i.e., $\mathfrak{I} = \{ F_\alpha : \alpha < \omega_c \}$.

Using Theorem 2, we construct a set B as follows: let $x_0, y_0 \in F_0$ where $x_0 \neq y_0$. Next, choose $x_1, y_1 \in F_1$ where $x_1 \neq y_1$ and both are different from x_0 and y_0. If $0 < \eta < \omega_c$ and if x_γ , y_γ have been defined for all $\gamma < \eta$, let

$$x_\eta, y_\eta \in F_\eta - \bigcup_{\gamma<\eta} \{ x_\gamma, y_\gamma \}$$

where $x_\eta \neq y_\eta$. Let

$$A_\eta = \{ x_\gamma : \gamma < \eta \} \cup \{ y_\gamma : \gamma < \eta \} = \bigcup_{\gamma<\eta} \{ x_\gamma, y_\gamma \}.$$

Note that $F_\eta = (F_\eta - A_\eta) \cup A_\eta$ is a disjoint union. Also, for all $\eta < \omega_c$, we have $|A_\eta| < \boldsymbol{c}$ since ω_c is the smallest ordinal number such that $| P_{\omega_c} | = \boldsymbol{c}$. Hence, by Corollary (4.30) in [1; p. 26], $|F_\eta - A_\eta| + |A_\eta| = |F_\eta| = |F_\eta| = \boldsymbol{c}$ implies that $| F_\eta - A_\eta | = \boldsymbol{c}$ for all $\eta < \omega_c$. Thus, the set $F_\eta - \bigcup_{\gamma<\eta} \{ x_\gamma, y_\gamma \}$ is not empty and has cardinality $\boldsymbol{c}$. Finally, let $B = \{ x_\eta : \eta < \omega_c \}$ and note that for all $\eta < \omega_c$, we have $B \cap F_\eta \neq \varnothing$ since $x_\eta \in B \cap F_\eta$. Similarly, we also have $(\mathbf{R} - B) \cap F_\eta \neq \varnothing$, since $y_\eta \in (\mathbf{R} - B) \cap F_\eta$.

We now prove that B is not λ_α - measurable if α is continuous and $\lambda_\alpha \neq 0$. Let K be a compact subset B; then K is closed since compact subsets of Hausdorff space are closed. Hence, K is countable, otherwise $K \in \mathfrak{I}$, which implies that $(\mathbf{R} - B) \cap K \neq \varnothing$. Similarly, if K^* is a compact subset of $\mathbf{R} - B$, then K^* is closed. Also K^* is countable, otherwise $B \cap K^* \neq \varnothing$. Thus, every compact subset of both B and $\mathbf{R} - B$ are countable.

Let α be any real-valued nondecreasing continuous function and let λ_α be the Lebesgue-Stieltjes measure on $\mathbf{R}$ where $\lambda_\alpha \neq 0$. Suppose that B is λ_α - measurable, then by Theorem 6

$$\lambda_\alpha(B) = \sup \{ \lambda_\alpha(K) : K \text{ is compact, } K \subset B \}.$$

Note that for each compact subset K of B, we have $K = \{ x_{\eta_1}, x_{\eta_2}, \ldots \}$ and thus,

$$\lambda_\alpha(K) = \lambda_\alpha\left(\bigcup_{k=1}^{\infty} \{ x_{\eta_k} \} \right) = \sum_{k=1}^{\infty} \lambda_\alpha(\{ x_{\eta_k} \}) = 0 .$$

Therefore, $\lambda_\alpha(B) = 0$ by the inner-regularity of λ_α. Similarly, for each compact subset K^* of $\mathbf{R} - B$, we have $K^* = \{ y_{\eta_1}, y_{\eta_2}, \ldots \}$ so that

$$\lambda_\alpha(K^*) = \lambda_\alpha\left(\bigcup_{k=1}^{\infty} \{ y_{\eta_k} \} \right) = \sum_{k=1}^{\infty} \lambda_\alpha(\{ y_{\eta_k} \}) = 0 .$$

By the inner-regularity of λ_α

$$\lambda_\alpha(\mathbf{R} - B) = \sup \{ \lambda_\alpha(K^*) : K^* \text{ is compact}, K^* \subset \mathbf{R} - B \} = 0.$$

Then we have $\lambda_\alpha(B) = 0 = \lambda_\alpha(\mathbf{R} - B)$. Since we have assumed that B is λ_α - measurable, we must have λ_α identically equal to zero measure. This is absurd since we assumed that $\lambda_\alpha \neq 0$. Therefore, the set $B = \{ x_\eta : \eta < \omega_c \}$ is *not* λ_α - measurable.

Note that λ_α is the Lebesgue measure on $\mathbf{R}$, if $\alpha(x) = x$. Also, for each open set $V \subset \mathbf{R}$ with $B \subset V$ we have, by Theorem 3,

$$\lambda_\alpha(B) = \inf \{ \lambda_\alpha(V) : V \text{ is open}, B \subset V \} > 0.$$

10 Remark. The construction of Bernstein sets can be carried out in a second countable, locally compact, Hausdorff space X with a measure μ which is diffused and regular on a σ - algebra $M \subset P(X)$ such that $\mu(X) > 0$. The construction is identical to the present construction. For a considerable generalization and insight, see [3; p. 133].

Every example of non λ - measurable set has been constructed by using the Axiom of Choice. It was announced by R. Solovay in the Notices of the American Mathematical Society **12**(1965) that without the Axiom of Choice, non Lebesgue measurable sets cannot be obtained at all.

References

[1] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag 1965

[2] J. Dugundji, *Topology*, Allyn and Bacon 1966

[3] W. F. Pfeffer, *Integrals and Measures*, Marcel and Dekker 1977