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Path Chromatic Number of Complementary Graphs

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The ***path chromatic number*** of a graph G, denoted by $\chi_\infty(G)$, is the minimum number of colors the points of G can be given so that each monochromatic color class induces a linear forest. Equivalently, it is the minimum number of subsets $V_1, V_2, \ldots, V_t$ into which $V(G)$ can be partitioned so that each $\langle V_i \rangle$ is a linear forest. A ***path partition*** of a graph G is a partition of the vertices of G such that each subset in the partition induces a linear forest.

This paper presents analogues of some interesting theorems due to Nordhaus and Gaddum [5] and John Mitchem [4].

Lemma 1. *Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ be a cycle, $n > 5$. The subgraph induced by any four points labeled consecutively in C_n is a P_4 in $\overline{C_n}$.*

Proof. Any four consecutive vertices in a cycle C_n, $n \geq 5$, form a path P_4 and P_4 is self-complementary. Hence, the same vertices also form a path in $\overline{C_n}$. □

Lemma 2. *Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ be a cycle, $n > 6$. The subgraph induced by at least five points of $V(\overline{C_n})$ contains a cycle.*

Proof. Given any five points of C_n, it is easy to see that there exist three points that are mutually non-adjacent. Thus, these three points are mutually adjacent in $\overline{C_n}$, forming a triangle. Hence, the subgraph induced by at least five points of $V(\overline{C_n})$ contains a cycle. □

Observation 1. Let P_n be a path of order n, $n \geq 4$. The subgraph induced by any four points labeled consecutively in P_n is a P_4 in $\overline{P_n}$.

Observation 2. The subgraph induced by at least five points of $V(\overline{P_n})$ contains a cycle, where P_n is path, $n \geq 5$.

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Theorem 1. *If C_n is a cycle of order n, $n \geq 3$, then* $\chi_\infty(\overline{C_n}) = \lceil n/4 \rceil$.

Proof. It is easy to check the theorem in the case $n = 3$, and 4. Let C_n be a cycle of order n, $n \geq 5$. Label $V(C_n)$ consecutively by $v_1, v_2, \ldots, v_n$. Note that $V(C_n) = V(\overline{C_n})$. From Lemma 2, the subgraph induced by at least five points in $\overline{C_n}$ contains a cycle. Thus, $\chi_\infty(\overline{C_n}) > n/5$. But from Lemma 1 any four consecutive points of $V(\overline{C_n})$ induces a path of order four. Thus, we color $V(\overline{C_n})$ by coloring every four consecutive vertices by a distinct color. Since the path chromatic number is an integer, we have $\chi_\infty(\overline{C_n}) = \lceil n/4 \rceil$. □

Theorem 2. *For cycles C_n, $n \geq 3$,*

(*i*) $\chi_\infty(C_n) + \chi_\infty(\overline{C_n}) > \chi_\infty(K_n)$ *if* $n < 7$;

(*ii*) $\chi_\infty(C_n) + \chi_\infty(\overline{C_n}) \leq \chi_\infty(K_n)$ *if* $n \geq 7$.

Proof. (*i*) Since $\chi_\infty(K_n) = \lceil n/2 \rceil$, $\chi_\infty(C_n) = 2$, and $\chi_\infty(\overline{C_n}) = \lceil n/4 \rceil$, we have

$$\lceil n/2 \rceil - \lceil n/4 \rceil = \lceil n/2 \rceil - \lceil (1/2)n/2 \rceil$$

$$\leq \lceil n/2 \rceil - 1/2 \lceil n/2 \rceil$$

$$= 1/2 \lceil n/2 \rceil < 2,$$

when $n < 7$. Thus $\lceil n/2 \rceil < 2 + \lceil n/4 \rceil$ when $n < 7$. Hence, $\chi_\infty(C_n \dot\cup \overline{C_n}) < \chi_\infty(C_n) + \chi_\infty(\overline{C_n})$ when $n < 7$.

(*ii*) if $n \geq 7$, it is easy to verify that $\lceil n/2 \rceil - \lceil n/4 \rceil \geq 2$. Thus, $\lceil n/2 \rceil \geq 2 + \lceil n/4 \rceil$. Therefore, $\chi_\infty(C_n \dot\cup \overline{C_n}) \geq \chi_\infty(C_n) + \chi_\infty(\overline{C_n})$ when $n \geq 7$. □

The following observations can be easily verified.

Theorem 3. *If P_n is a path of order n, $n \geq 1$, then* $\chi_\infty(\overline{P_n}) = \lceil n/4 \rceil$.

Proof. The proof is similar to that of Theorem 1. □

Theorem 4. *For complete bipartite graphs $K_{m,n}$,*

$$\chi_\infty(\overline{K}_{m,n}) = \max\{\chi_\infty(K_m), \chi_\infty(K_n)\}.$$

Proof. Let $K_{m,n}$ be a complete bipartite graph. Then the complement of $K_{m,n}$ denoted by $\overline{K}_{m,n}$ has two components, which are complete graphs K_m and K_n.

Clearly, the path chromatic number of $\overline{K}_{m,n}$ is the maximum of the path chromatic numbers between K_m and K_n. □

Theorem 5. *For complete bipartite graphs $K_{m,n}$, $m \geq n$,*

(*i*) $\chi_\infty\left(K_{m,n}\right)+\chi_\infty\left(\overline{K}_{m,n}\right)>\chi_\infty\left(K_{m+n}\right)$ if $n \geq 2$, or $n = 3$ and m is odd.

(*ii*) $\chi_\infty\left(K_{m,n}\right)+\chi_\infty\left(\overline{K}_{m,n}\right)=\chi_\infty\left(K_{m+n}\right)$ *if $n = 3$ and m is even, or $n = 4$, or $n = 5$ and m is odd.*

(*iii*) $\chi_\infty\left(K_{m,n}\right)+\chi_\infty\left(\overline{K}_{m,n}\right)<\chi_\infty\left(K_{m+n}\right)$ if $n = 5$ and m is even, or $n \geq 6$.

Proof. $K_{m,n} \dot{\cup} \overline{K}_{m,n} = K_{m+n}$; $\chi_\infty\left(K_{m+n}\right)=\lceil(m+n)/2\rceil$ by [1];

$$\chi_\infty\left(K_{m,n}\right)=2 \text{ and } \chi_\infty\left(\overline{K}_{m,n}\right)=\lceil m/2\rceil$$

from Theorem 4 and $m \geq n$. Consider $\lceil(m+n)/2\rceil-\lceil m/2\rceil$.

Case 1. $\lceil(m+n)/2\rceil-\lceil m/2\rceil<2$.

Subcase 1.1. m is odd, and n is even. In this case we have $(m + n)/2 + 1/2 - m/2 - 1/2 < 2$. Hence, $n/2 < 2$ and thus $n < 4$.

Subcase 1.2. m is even, and n is even. Then $(m + n)/2 - m/2 < 2$. Hence, $n/2 < 2$ and thus $n < 4$.

Subcase 1.3. m is even, and n is odd. Then $(m + n)/2 + 1/2 - m/2 < 2$. Hence, $n/2 < 3/2$ and thus $n < 3$.

Subcase 1.4. m is odd, and n is odd. Then $(m + n)/2 - m/2 - 1/2 < 2$. Hence, $n/2 < 5/2$ and thus $n < 5$.

Thus, $\lceil(m+n/2)\rceil-\lceil n/2\rceil<2$ if $n \leq 2$, or $n = 3$ and m is odd. Therefore,

$$\lceil(m+n)/2\rceil<2+\lceil m/2\rceil.$$

Hence,

$$\chi_\infty(K_{m,n} \dot{\cup} K_{m,n})<\chi_\infty\left(\overline{K}_{m,n}\right)+\chi_\infty(\overline{K}_{m,n})$$

if $n \leq 2$, or $n = 3$ and m is odd.

Case 2. $\lceil (m+n)/2 \rceil - \lceil m/2 \rceil = 2$. Using a similar argument as in case 1, we also have the following subcases:

Subcase 2.1. When m is odd, and n is even, then $n = 4$;

Subcase 2.2. When m is even, and n is even, then $n = 4$;

Subcase 2.3. When m is even, and n is odd, then $n = 3$;

Subcase 2.4. When m is odd, and n is odd, then $n = 5$.

Thus, $\lceil (m+n)/2 \rceil - \lceil m/2 \rceil = 2$, if $n = 4$, or $n = 3$ and m is even, or $n = 5$ and m is odd. Therefore,

$$\lceil (m+n)/2 \rceil = 2 + \lceil m/2 \rceil.$$

Hence, $\chi_\infty(K_{m,n} \dot{\cup} \overline{K}_{m,n}) = \chi_\infty(K_{m,n}) + \chi_\infty(\overline{K}_{m,n})$ if $n = 4$, or $n = 3$ and m is even, or $n = 5$ and m is odd.

Case 3. $\lceil (m+n)/2 \rceil - \lceil m/2 \rceil > 2$. Again, using an argument similar to that in case 1, we have:

Subcase 3.1. When m is odd, and n is even, then $n < 4$;

Subcase 3.2. When m is even, and n is even, then $n < 4$;

Subcase 3.3. When m is even, and n is odd, then $n < 3$;

Subcase 3.4. When m is odd, and n is odd, then $n < 5$.

Thus, $\lceil (m+n)/2 \rceil - \lceil m/2 \rceil > 2$ if $n \geq 6$, or $n = 5$ and m is even. Hence,

$$\lceil (m+n)/2 \rceil > 2 + \lceil m/2 \rceil.$$

Hence, $\chi_\infty(K_{m,n} \dot{\cup} \overline{K}_{m,n}) > \chi_\infty(K_{m,n}) + \chi_\infty(\overline{K}_{m,n})$ if $n \geq 6$, or $n = 5$ and m is even. □

Corollary 1. *For* $K_{m,1}$, $\chi_\infty(K_{m,1} \dot{\cup} \overline{K}_{m,1}) < \chi_\infty(\overline{K}_{m,1})$.

Proof. This falls under case 1 of Theorem 5(*iii*). □

Two of the many results dealing with the chromatic numbers of graphs, in particular, is the chromatic number of a graph and its complement due to Nordhaus and Gaddum [5]. They are given in the next two theorems.

Theorem 6. *If G is a graph of order p and $\chi(G)$ and $\chi(\overline{G})$ denote respectively the chromatic numbers of a graph G and its complement $\overline{G}$, then*

(*i*) $2\sqrt{p} \le \chi(G) + \chi(\overline{G}) \le p + 1$;

(*ii*) $p \le \chi(G)\chi(\overline{G}) \le [(p+1)/2]^2$.

John Mitchem presented an analogous result in [4] involving point arboricity. The ***point arboricity*** $\rho(G)$ of a graph G is the minimum number of subsets into which the vertices of G can be partitioned so that each subset induces an acyclic subgraph. His result may now be stated:

Theorem 7. *If G is a graph of order p, then*

(*i*) $\sqrt{p} \le \rho(G) + \rho(\overline{G}) \le (p+3)/2$;

(*ii*) $p/4 \le \rho(G) + \rho(\overline{G}) \le [(p+3)/4]^2$.

Looking at Theorems 6 and 7, we can now formulate analogous results for the path chromatic number. First, let us observe that if G is a graph such that $\chi_\infty(G) = t$, then $V(G)$ can be partitioned into t subsets such that each subset induces a linear forest. Now, since these are acyclic graphs,

$$\rho(G) \le \chi_\infty(G).$$

Note that the path chromatic number of an acyclic graph does not exceed two, thus $\chi_\infty(G) \le 2\rho(G)$. Hence,

$$\chi_\infty(G)/2 \le \rho(G) \le \chi_\infty(G).$$

It is easy to see that these bounds for $\rho(G)$ are best possible. If $G = K_{1,n}$ $(n > 1)$, then $\chi_\infty(G) = 2$ and $\rho(G) = 2/2 = 1$. If $G = P_n$, then $\chi_\infty(G) = 1 = \rho(G)$.

Theorem 8. *For $t \ge 2$, let $G_i = (V_i, E_i)$, $i = 1, 2, \ldots, t$ be mutually disjoint linear forests with $2 \le |V_1| \le |V_2| \le \cdots \le |V_t|$ and $|E_i| \ge 1$, $i = 1, 2, \ldots, t$.*

(*i*) *Then $\chi_\infty(G) = t$ where $G = \sum_{i=1}^{t} G_i$;*

(*ii*) *If each G_i is a path then $\chi_\infty(\overline{G}) = \lceil p/4 \rceil$, where $p = max\{|G_i|\}$.*

Proof. (*i*) Let H be the induced subgraph of G containing two adjacent points from each G_i.

Then $H = K_{2t}$ and $\chi_\infty(K_{2t}) = t$. Note that $\chi_\infty(H) \leq \chi_\infty(G)$. Clearly, $V_1, V_2, \ldots, V_t$ is a path partition of $V(G)$ so that $\chi_\infty(G) \leq t$. Thus $t \leq \chi_\infty(G) \leq t$ which implies $\chi_\infty(G) = t$. We are left to prove (*ii*). $\overline{G}$ has at least t components which are complements of paths. The component of $\overline{G}$ with maximum number of edges is the complement of a G_i with maximum order p, i.e., G_t. Since G_t is a path, from Theorem 2 we know that $\chi_\infty(\overline{G_t}) = \lceil p/4 \rceil$, where p is the order of G_t. But the path chromatic number of a graph is the path chromatic number of its largest component. Therefore,

$$\chi_\infty(\overline{G}) = \lceil p/4 \rceil. \quad \square$$

Theorem 9. *Let G be a graph of order p, then*

(*i*) $\sqrt{p} \leq \chi_\infty(G) + \chi_\infty(\overline{G}) \leq p + 1$;

(*ii*) $p/4 \leq \chi_\infty(G)\chi_\infty(\overline{G}) \leq [(p+1)/2]^2$.

Proof. Theorem 7 and Lemma 5.4 in [2] imply the lower bounds of (*i*) and (*ii*). Theorem 6 implies the upper bounds since $\chi_\infty(G) \leq \chi(G)$. $\square$

The lower bound $\sqrt{p}$ for $\chi_\infty(G) + \chi_\infty(\overline{G})$ is best possible. To see this, let $G_i = P_{16}$, $i = 1, 2, 3, 4$ and let $\sum_{i=1}^{4} G_i$. By Proposition 5.1 in [2], $\chi_\infty(G) = 4$ and $\chi_\infty(\overline{G}) = \lceil 16/4 \rceil = 4$. Thus, $\chi_\infty(G) + \chi_\infty(\overline{G}) = 4 + 4 = 8 = \sqrt{16 \times 4} = \sqrt{p}$. Note that $\chi_\infty(G)\chi_\infty(\overline{G})$ is also the best possible.

The upper bound for Theorem 9(*i*) may be improved.

Theorem 10. *Let G be a graph of order p, then*

$$\chi_\infty(G) + \chi_\infty(\overline{G}) \leq p.$$

Proof. Case 1. $G = K_p$. From [1], $\chi_\infty(G) = \lceil p/2 \rceil$ and $\chi_\infty(\overline{G}) = 1$. Therefore,

$$\chi_\infty(G) + \chi_\infty(\overline{G}) = \lfloor p/2 \rfloor + \lceil p/2 \rceil = p.$$

Case 2. $G \neq K_p$. This implies that G is a subgraph of $K_p - e$, where e is an edge of K_p. Thus $\chi_\infty(G) \leq \chi_\infty(K_p - e) = \lfloor p/2 \rfloor$. Also, since $\overline{G}$ is of order p, then $\chi_\infty(G) \leq \lceil p/2 \rceil$. Therefore, $\chi_\infty(G) + \chi_\infty(\overline{G}) \leq \lfloor p/2 \rfloor + \lceil p/2 \rceil = p$. □

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